Question 1: Evaluate $\int_{-\infty}^{0} xe^x\,dx$ making proper use of required limits.

$$\int_{-\infty}^{0} xe^x\,dx$$

$$= \lim_{a\to-\infty} \int_a^0 xe^x\,dx$$

- For $\int xe^x\,dx$:

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$\therefore \int xe^x dx = \int u\,dv$

$= uv - \int v\,du$

$= xe^x - \int e^x dx$

$= xe^x - e^x + C.$

$$= \lim_{a\to-\infty} \left[xe^x - e^x\right]_a^0$$

$$= \lim_{a\to-\infty} \left[(0-1) - (ae^a - e^a)\right]$$

$$= \lim_{a\to-\infty} \left[-1 - \underbrace{ae^a}_{\to 0} - \underbrace{e^a}_{\to 0}\right]$$

$$= \boxed{-1}$$

- $\lim_{a\to-\infty} ae^a \sim \text{"}-\infty\cdot 0\text{"}$

$= \lim_{a\to-\infty} \frac{a}{e^{-a}} \sim \frac{-\infty}{\infty}$

$\overset{H}{=} \lim_{a\to-\infty} \frac{1}{-e^{-a}} = 0$

[5]

Question 2: Use the comparison theorem to determine whether $\int_1^\infty \frac{\sin^2(x)}{x^2+x^4}\,dx$ converges or diverges.

$$0 \le \frac{\sin^2(x)}{x^2+x^4} \le \frac{1}{x^2}$$

Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges (p-integral, $p = 2 > 1$),

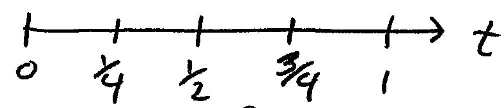
so does $\int_1^\infty \frac{\sin^2(x)}{x^2+x^4}\,dx$ by the comparison theorem.

[5]

Question 3:

(a) Use Simpson's rule on 4 subintervals to approximate $\int_0^1 \sin(\pi t)\,dt$. (Simplify your final answer.)

$\Delta t = \frac{1-0}{4} = \frac{1}{4}$, $f(t) = \sin(\pi t)$;



$$S_4 = \frac{\Delta t}{3}\left[f(0) + 4f(\tfrac{1}{4}) + 2f(\tfrac{1}{2}) + 4f(3/4) + f(1)\right]$$

$$= \frac{(1/4)}{3}\left[\underbrace{\sin(0)}_{0} + 4\sin(\tfrac{\pi}{4}) + 2\underbrace{\sin(\tfrac{\pi}{2})}_{1} + 4\sin(\tfrac{3\pi}{4}) + \underbrace{\sin(\pi)}_{0}\right]$$

$$= \frac{1}{12}\left[4(\tfrac{1}{\sqrt{2}}) + 2(1) + 4(\tfrac{1}{\sqrt{2}})\right]$$

$$= \frac{1}{12}\left(\frac{8}{\sqrt{2}} + 2\right) = \left(\frac{\cancel{8}^{4}\sqrt{2} + \cancel{4}^{2}}{\cancel{2}}\right)\left(\frac{1}{12}\right) = \frac{\cancel{4}^{2}\sqrt{2} + \cancel{2}^{1}}{\cancel{12}_{6}} = \boxed{\frac{2\sqrt{2}+1}{6}}$$

[5]

(b) Determine an error bound $|E_{S_4}|$ on your approximation in part (a). You may leave your answer in a calculator ready form.

(Recall: the error in using Simpson's rule to approximate $\int_a^b f(x)\,dx$ using n subintervals is at most $\frac{K(b-a)^5}{180n^4}$ where $|f^{(4)}(x)| \le K$ on $[a, b]$.)

$f'(t) = \pi\cos(\pi t)$; $f''(t) = -\pi^2\sin(\pi t)$; $f'''(t) = -\pi^3\cos(\pi t)$;

$f^{(4)}(t) = \pi^4\sin(\pi t)$.

$$|f^{(4)}(t)| = |\pi^4\sin(\pi t)| \le \pi^4 \text{ on } [0,1].$$

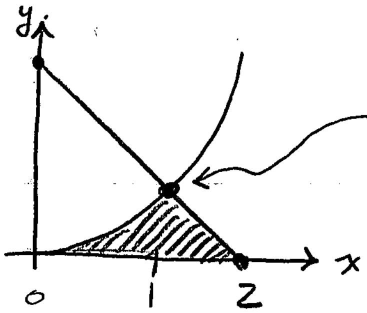
$\therefore\ K = \pi^4.$

$$\therefore\ |E_{S_4}| \le \frac{\pi^4(1-0)^5}{180\,(4)^4} = \boxed{\frac{(\pi/4)^4}{180}}$$

[5]

Question 4:

(a) Determine the area of the region in the first quadrant that is bounded by the curves $y = x^2$, $y = 2 - x$ and $y = 0$.



$x^2 = 2 - x$
$\Rightarrow x^2 + x - 2 = 0$
$\Rightarrow (x-1)(x+2) = 0$
$\Rightarrow x = 1, -2$

$\therefore A = \int_0^1 x^2 - 0 \, dx + \int_1^2 (2-x) - 0 \, dx$

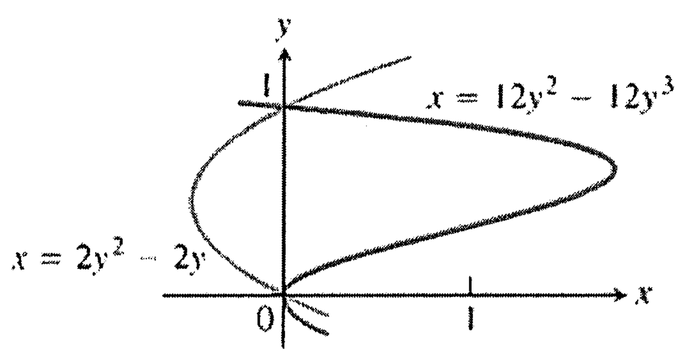
$= \left[\frac{x^3}{3}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$

$= \frac{1}{3} + \cancel{4} - \cancel{2} - \cancel{2} + \frac{1}{2}$

$= \boxed{\frac{5}{6}}$

[5]

(b) Determine the area of the shaded region:



$A = \int_{y=0}^{1} (12y^2 - 12y^3) - (2y^2 - 2y) \, dy$

$= \left[\frac{12}{3} y^3 - \frac{12}{4} y^4 - \frac{2}{3} y^3 + \frac{2}{2} y^2\right]_0^1$

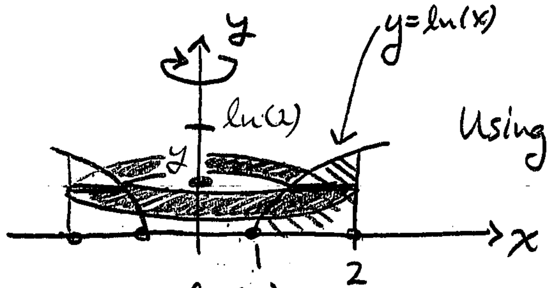
$= 4 - 3 - \frac{2}{3} + 1$

$= \boxed{\frac{4}{3}}$

[5]

Question 5:

(a) The region in the first quadrant that is bounded by $y = \ln x$, $y = 0$ and $x = 2$ is rotated about the y-axis. Determine the volume of the resulting solid.



$y = \ln(x) \iff x = e^y.$

Using horizontal washers:

$$V = \int_{y=0}^{\ln(2)} \pi\, 2^2 - \pi (e^y)^2 \, dy$$

$$= \int_0^{\ln(2)} 4\pi - \pi e^{2y} \, dy$$

$$= \left[4\pi y - \frac{\pi}{2} e^{2y} \right]_0^{\ln(2)}$$

$$= 4\pi \ln(2) - \frac{\pi}{2} e^{2\ln(2)} - 0 + \frac{\pi}{2}$$

$$= 4\pi \ln(2) - \frac{\pi}{2} e^{\ln(2^2)} + \frac{\pi}{2}$$

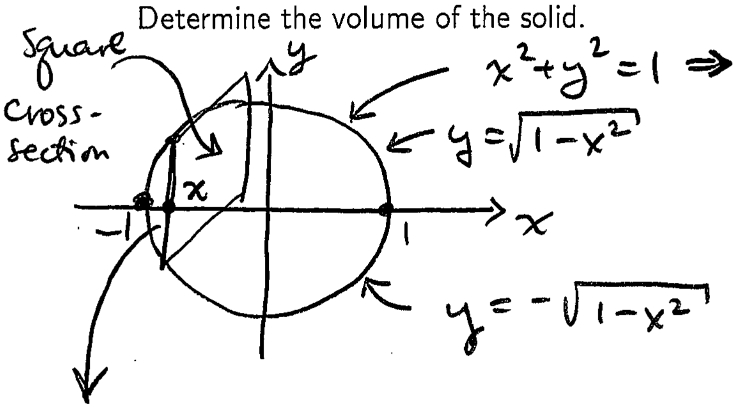
$$= 4\pi \ln(2) - \frac{4\pi}{2} + \frac{\pi}{2}$$

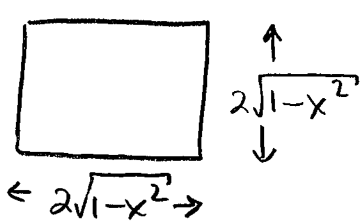
$$= \boxed{\frac{\pi}{2}\left(8\ln(2) - 3\right)}$$

[5]

(b) The flat base of a solid is a circle of radius 1. Parallel cross-sections perpendicular to the base are squares. Determine the volume of the solid.

$x^2 + y^2 = 1 \Rightarrow y = \pm\sqrt{1-x^2}$





$$\therefore A(x) = \left(2\sqrt{1-x^2}\right)^2 = 4(1-x^2)$$

$$\therefore V = \int_{-1}^{1} 4(1-x^2)\, dx$$

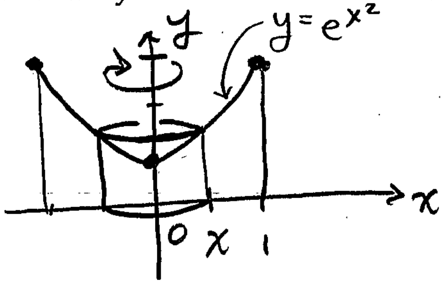
$$= 2\int_0^1 4 - 4x^2 \, dx$$

$$= 2\left[4x - \frac{4x^3}{3}\right]_0^1 = 2\left(4 - \frac{4}{3}\right) = \boxed{\frac{16}{3}}$$

[5]

Question 6:

(a) The region in the first quadrant that is bounded by the curves $y = e^{x^2}$, $y = 0$, $x = 0$ and $x = 1$ is rotated about the y-axis. Determine the volume of the resulting solid.



Using (vertical) cylinders:

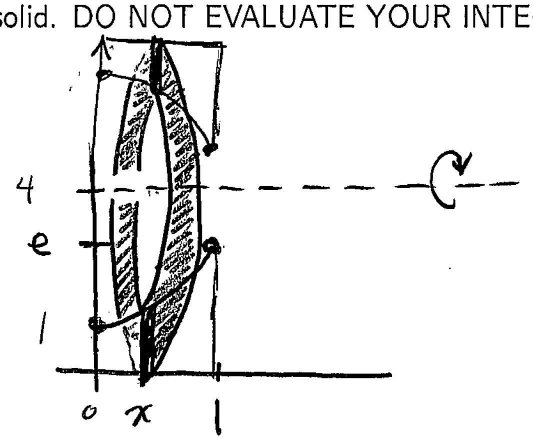
$$\therefore V = \int_0^1 2\pi x\, e^{x^2}\, dx \quad \left\}\begin{aligned} u &= x^2 \\ du &= 2x\,dx \end{aligned}\right.$$

$$= \pi \left[e^{x^2} \right]_0^1$$

$$= \boxed{\pi(e-1)}$$

[5]

(b) The region in the first quadrant that is bounded by the curves $y = e^{x^2}$, $y = 0$, $x = 0$ and $x = 1$ is rotated about the line $y = 4$. Set up an expression using integrals which represents the volume of the resulting solid. DO NOT EVALUATE YOUR INTEGRALS.



$$y = e^{x^2} \Leftrightarrow x = \sqrt{\ln(y)}.$$

By washers:

$$V = \int_{x=0}^{1} \pi(4^2) - \pi(4 - e^{x^2})^2\, dx$$

By Cylindrical Shells:

$$V = \int_{y=0}^{1} 2\pi(4-y)\,1\,dy + \int_{y=1}^{e} 2\pi(4-y)(1-\sqrt{\ln y})\,dy$$

[5]