Question 1: Determine $\int x^2 e^{2x}\,dx = I$

Let $u = x^2$ $dv = e^{2x}dx$

$du = 2x\,dx$ $v = \frac{1}{2}e^{2x}$

$\therefore I = \int u\,dv = uv - \int v\,du$

$= \frac{1}{2}x^2e^{2x} - \underbrace{\int \cancel{\tfrac{1}{2}} e^{2x} \cdot \cancel{2}x\,dx}$

let $u = x$ $dv = e^{2x}dx$

$du = dx$ $v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}x^2e^{2x} - \left[\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx\right]$

$= \boxed{\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C}$

[5]

Question 2: Determine $\int \arcsin(x)\,dx = I$

Let $u = \arcsin(x)$ $dv = dx$

$du = \frac{1}{\sqrt{1-x^2}}dx$ $v = x$

$\therefore I = \int u\,dv = uv - \int v\,du$

$= x\arcsin(x) - \left(\frac{-1}{2}\right)\underbrace{\int \frac{(-2)x}{\sqrt{1-x^2}}\,dx}$

let $u = 1-x^2$, $du = -2x\,dx$

$= x\arcsin(x) + \frac{1}{2}\int u^{-1/2}du$

$= x\arcsin(x) + u^{1/2} + C$

$= \boxed{x\arcsin(x) + \sqrt{1-x^2} + C}$

[5]

Question 3: Determine $\int_0^{\pi/4} \tan^4(t)\,dt$

$$I = \int \tan^4(t)\,dt = \int \tan^2(t)\tan^2(t)\,dt$$

$$= \int \tan^2(t)(\sec^2(t)-1)\,dt$$

$$= \int \tan^2(t)\sec^2(t)\,dt - \int \tan^2(t)\,dt$$

$$= \underbrace{\int \tan^2(t)\sec^2(t)\,dt}_{u=\tan(t),\ du=\sec^2(t)dt} - \int \sec^2(t)-1\,dt$$

$$= \int u^2\,du - [\tan(t)-t]$$

$$= \frac{1}{3}\tan^3(t) - \tan(t) + t + C$$

$$\therefore \int_0^{\pi/4} \tan^4(t)\,dt = \left[\frac{1}{3}\tan^3(t)-\tan(t)+t\right]_0^{\pi/4} = \left[\frac{1}{3}\cdot 1^3 - 1 + \frac{\pi}{4}\right] - [0-0+0] = \boxed{\frac{\pi}{4}-\frac{2}{3}}$$ [5]

Question 4: Determine $\int \sin^3\theta\cos^4\theta\,d\theta = I$

$$I = \int \sin^2\theta\cos^4\theta\sin\theta\,d\theta$$

$$= -\int (1-\cos^2\theta)\cos^4\theta(-\sin\theta)\,d\theta \quad \left.\right\} \begin{array}{l}\text{let } u=\cos\theta\\ du = -\sin\theta\,d\theta\end{array}$$

$$= -\int (1-u^2)u^4\,du$$

$$= -\frac{u^5}{5} + \frac{u^7}{7} + C$$

$$= \boxed{-\frac{\cos^5\theta}{5} + \frac{\cos^7\theta}{7} + C}$$

[5]

Question 5: Determine $\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = I$

Let $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$$\therefore I = \int \frac{1}{(2\sin\theta)^2\sqrt{4-(2\sin\theta)^2}}\, 2\cos\theta\, d\theta$$

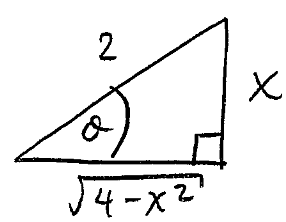
$$= \int \frac{1}{4\sin^2\theta\sqrt{4(1-\sin^2\theta)}} \cdot 2\cos\theta\, d\theta$$

$$= \int \frac{1}{\underset{4}{\cancel{8}}\sin^2\theta\, \cancel{\cos\theta}}\, \cancel{2\cos\theta}\, d\theta$$

$x = 2\sin\theta$

$\Rightarrow \sin\theta = \frac{x}{2}$

$$= \frac{1}{4}\int \csc^2\theta\, d\theta$$



$$= -\frac{1}{4}\cot\theta + C$$

$$= \boxed{-\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C}$$

[10]

Question 6: Determine $\int \frac{x^2-x+6}{x(x^2+3)}\,dx = I$

$$\frac{x^2-x+6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

$$= \frac{Ax^2+3A+Bx^2+Cx}{x(x^2+3)}$$

$$= \frac{(A+B)x^2+Cx+3A}{x(x^2+3)}$$

$\therefore$ ① $A+B=1$, ② $C=-1$, ③ $3A=6$

① $\Rightarrow C=-1$

③ $\Rightarrow A=2$

① $\Rightarrow B=1-A=-1$

$$\therefore I = \int \frac{2}{x} + \frac{-x-1}{x^2+3}\,dx$$

$$= \int \frac{2}{x}\,dx - \frac{1}{2}\int \frac{2x}{x^2+3}\,dx - \int \frac{1}{x^2+(\sqrt{3})^2}\,dx$$

$$= 2\int \frac{1}{x}\,dx - \frac{1}{2}\underbrace{\int \frac{2x}{x^2+3}\,dx}_{\substack{u=x^2+3\\ du=2x\,dx}} - \frac{1}{\sqrt{3}}\underbrace{\int \frac{1/\sqrt{3}}{\left(\frac{x}{\sqrt{3}}\right)^2+1}\,dx}_{w=\frac{x}{\sqrt{3}},\ dw=\left(\frac{1}{\sqrt{3}}\right)dx}$$

$$= 2\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{1}{u}\,du - \frac{1}{\sqrt{3}}\int \frac{1}{1+w^2}\,dw$$

$$= 2\ln|x| - \frac{1}{2}\ln|u| - \frac{1}{\sqrt{3}}\arctan(w) + C$$

$$= \boxed{2\ln|x| - \frac{1}{2}\ln|x^2+3| - \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + C}$$

[10]

Question 7: Determine $\int \frac{1}{x^2+4x+7}\,dx = I$

$$x^2+4x+7 = (x+2)^2+3$$

$$\therefore\ I = \int \frac{1}{(x+2)^2+3}\,dx$$

$$= \frac{1}{\sqrt{3}} \int \frac{1/\sqrt{3}}{\left(\frac{x+2}{\sqrt{3}}\right)^2+1}\,dx$$

let $u = \frac{x+2}{\sqrt{3}}$

$du = \frac{1}{\sqrt{3}}dx$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{1+u^2}\,du$$

$$= \frac{1}{\sqrt{3}} \arctan(u) + C$$

$$= \boxed{\frac{1}{\sqrt{3}} \arctan\left(\frac{x+2}{\sqrt{3}}\right) + C}$$

[5]

Question 8: Determine $\int \frac{x^3}{\sqrt{x^2+4}}\,dx = I$

Let $u = x^2+4 \Rightarrow x^2 = u-4$

$du = 2x\,dx$

$$\therefore\ I = \frac{1}{2}\int \frac{x^2\,(2x)}{\sqrt{x^2+4}}\,dx$$

$$= \frac{1}{2}\int \frac{u-4}{\sqrt{u}}\,du$$

$$= \frac{1}{2}\int u^{1/2} - 4u^{-1/2}\,du$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{3/2} - 4\cdot 2u^{1/2}\right] + C$$

$$= \boxed{\frac{1}{3}(x^2+4)^{3/2} - 4(x^2+4)^{1/2} + C}$$

[5]