Question 1: Use the definition of the definite integral in the form

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

to evaluate

$$\int_0^2 (x - x^3)\,dx$$

Carefully set up the Riemann sum and clearly show the steps of your simplification.

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

$$x_i = a + i\Delta x = 0 + i\left(\frac{2}{n}\right) = \frac{2i}{n}$$

$$f(x_i) = x_i - (x_i)^3 = \frac{2i}{n} - \left(\frac{2i}{n}\right)^3 = \frac{2i}{n} - \frac{8i^3}{n^3}.$$

$$\int_0^2 (x - x^3)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{2i}{n} - \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{4i}{n^2} - \frac{16i^3}{n^4}\right]$$

$$= \lim_{n\to\infty} \left[\frac{4}{n^2}\left(\sum_{i=1}^{n} i\right) - \frac{16}{n^4}\left(\sum_{i=1}^{n} i^3\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{4}{n^2}\,\frac{n(n+1)}{2} - \frac{16}{n^4}\,\frac{n^2(n+1)^2}{4}\right]$$

$$= \lim_{n\to\infty} \left[2 \cdot \frac{n}{n} \cdot \frac{n+1}{n} - 4 \cdot \frac{n^2}{n^2} \cdot \frac{(n+1)^2}{n^2}\right]$$

$$= 2 - 4$$

$$= \boxed{-2}$$

[10]

Question 2: Determine the average value of $f(x) = e^{\cos(x)} \sin(x)$ over the interval $[0, \pi/2]$.

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{\frac{\pi}{2}-0}\int_0^{\frac{\pi}{2}} e^{\cos(x)} \sin(x)\,dx$$

$$\left.\begin{array}{l} \text{Let } u=\cos(x) \\ du=-\sin(x)\,dx \\ x=0 \Rightarrow u=1 \\ x=\frac{\pi}{2} \Rightarrow u=0 \end{array}\right.$$

$$= -\frac{2}{\pi}\int_1^0 e^u\,du$$

$$= \frac{2}{\pi}\int_0^1 e^u\,du$$

$$= \frac{2}{\pi}\left[e^u\right]_0^1 = \boxed{\frac{2}{\pi}(e-1)}$$

[5]

Question 3: A town's population is changing at a rate of $30t(2-t)$ people per year where $t=0$ corresponds to the present. Determine the population size at the end of two years if the population is currently 100.

$$P(2) - P(0) = \int_0^2 30t(2-t)\,dt$$

$$= 30\int_0^2 2t - t^2\,dt$$

$$= 30\left[\frac{\cancel{2}t^2}{\cancel{2}} - \frac{t^3}{3}\right]_0^2$$

$$= 30\left(4 - \frac{8}{3}\right)$$

$$= 40$$

$$\therefore\ P(2) = P(0) + 40 = 100 + 40 = \boxed{140 \text{ people}}$$

[5]

Question 4: Determine the following integrals:

(a) $\int \frac{1+x+\sqrt{x}}{x^2}\,dx$

$= \int x^{-2} + \frac{1}{x} + x^{-3/2}\,dx$

$= \frac{x^{-1}}{-1} + \ln|x| + \frac{x^{-1/2}}{(-1/2)} + C$

$= \boxed{\frac{-1}{x} + \ln|x| - \frac{2}{\sqrt{x}} + C}$

[3]

(b) $\frac{1}{3}\int 3x^2\sqrt{x^3+4}\,dx$ $\left\{ \begin{array}{l} \text{let } u = x^3+4 \\ du = 3x^2\,dx \end{array} \right.$

$= \frac{1}{3}\int u^{1/2}\,du$

$= \frac{1}{3}\,\frac{u^{3/2}}{(3/2)} + C$

$= \boxed{\frac{2}{9}(x^3+4)^{3/2} + C}$

[4]

(c) $\int \frac{2x+5}{x^2+5x-14}\,dx$ $\left\{ \begin{array}{l} \text{let } u = x^2+5x-14 \\ du = (2x+5)\,dx \end{array} \right.$

$= \int \frac{1}{u}\,du$

$= \ln|u| + C$

$= \boxed{\ln|x^2+5x-14| + C}$

[3]

Question 5: Determine the following integrals:

(a) $\int_e^{e^4} \frac{1}{x\sqrt{\ln(x)}}\,dx$ let $u = \ln(x)$; $x = e \Rightarrow u = \ln(e) = 1$

$du = \frac{1}{x}\,dx$; $x = e^4 \Rightarrow u = \ln(e^4) = 4$

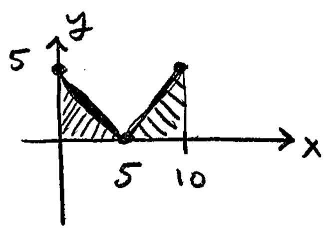
$= \int_1^4 \frac{1}{\sqrt{u}}\,du$

$= \left[2\sqrt{u}\right]_1^4$

$= 2\sqrt{4} - 2\sqrt{1} = \boxed{2}$

[3]

(b) $\int_0^{10} |x-5|\,dx = 2 \cdot \frac{1}{2} \cdot 5 \cdot 5 = \boxed{25}$



[3]

(c) $\int_0^1 \frac{e^x}{1+e^{2x}}\,dx$ $\Big\}$ $u = e^x$, $du = e^x\,dx$ $\Big\}$ $x = 0 \Rightarrow u = 1$, $x = 1 \Rightarrow u = e$

$= \int_1^e \frac{1}{1+u^2}\,du$

$= \left[\arctan(u)\right]_1^e$

$= \arctan(e) - \arctan(1)$

$= \boxed{\arctan(e) - \frac{\pi}{4}}$

[4]

Question 6: Find both the number a and the function f such that

$$6 + \int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x}$$

for every $x > 0$. (Hint: differentiate both sides of the equation.)

Let $x = a$: $\quad 6 + \underbrace{\cancel{\int_a^a \frac{f(t)}{t^2}}}_{0} = 2\sqrt{a}$

$$\Rightarrow \quad 3 = \sqrt{a}$$

$$\Rightarrow \quad \boxed{a = 9}$$

$$\frac{d}{dx}\left[6 + \int_9^x \frac{f(t)}{t^2}\,dt\right] = \frac{d}{dx}\left[2\sqrt{x}\right]$$

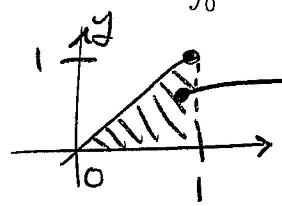
$$\Rightarrow \quad 0 + \frac{f(x)}{x^2} = \frac{1}{\sqrt{x}}$$

$$\Rightarrow \quad f(x) = \frac{x^2}{\sqrt{x}} = \boxed{x^{3/2}}$$

[5]

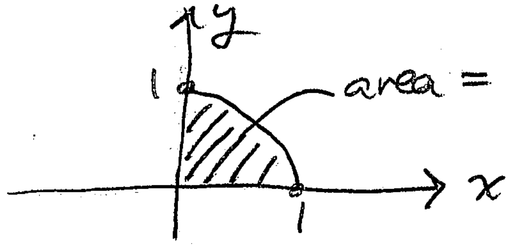
Question 7: Use an area interpretation to determine $\int_0^1 x + \sqrt{1 - x^2}\,dx$

- $y = x$ has graph



area $= (\frac{1}{2})(1)(1) = \frac{1}{2}$

- $y = \sqrt{1-x^2}$ over $[0,1]$ is the sector of the circle $x^2 + y^2 = 1$ in 1^{st} quadrant:



area $= \frac{1}{4} \cdot \pi \cdot 1^2 = \frac{\pi}{4}$

$\therefore \int_0^1 x + \sqrt{1-x^2}\,dx$ = total area $= \boxed{\frac{2+\pi}{4}}$

[5]