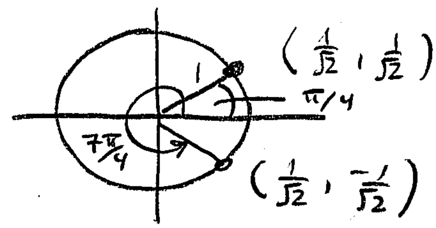
Question 1:

(a) Determine $\sin\left(\sin^{-1}(3/4)\right)$ $= \boxed{\frac{3}{4}}$

[2]

(b) Determine $\arccos\left(\cos\left(7\pi/4\right)\right)$

$\cos\left(\frac{7\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) \longleftrightarrow$



$\therefore \arccos\left(\cos\left(\frac{7\pi}{4}\right)\right)$

$= \arccos\left(\cos\left(\frac{\pi}{4}\right)\right)$

$= \boxed{\frac{\pi}{4}}$ since $0 \leq \frac{\pi}{4} \leq \pi$

[4]

(c) Let $f(x) = x\tan^{-1}(x) - \dfrac{1}{2}\ln\left(1+x^2\right)$. Evaluate $f'(1)$.

$f'(x) = 1 \cdot \tan^{-1}(x) + \cancel{\frac{x}{1+x^2}} - \cancel{\frac{1}{2} \cdot \frac{1}{1+x^2} \cdot 2x}$

$= \tan^{-1}(x).$

$\therefore f'(1) = \tan^{-1}(1) = \boxed{\frac{\pi}{4}}$

[4]

Question 2:

(a) Find all values of x at which the tangents to $f(x) = \sinh^2(x)$ have the same slope as the tangents to $g(x) = \cosh(x)$.

Solve $f'(x) = g'(x)$ for x

$\Rightarrow 2\sinh(x)\cosh(x) = \sinh(x)$

$\Rightarrow 2\sinh(x)\cosh(x) - \sinh(x) = 0$

$\Rightarrow \sinh(x)\left[2\cosh(x) - 1\right] = 0$

$\Rightarrow \sinh(x) = 0$, $2\cosh(x) - 1 = 0$

$\Rightarrow \boxed{x = 0}$, $\underbrace{\cosh(x) = \frac{1}{2}}$

no solutions since $\cosh(x) \geq 1$ for every x

[5]

(b) Evaluate the following limit: $\lim\limits_{x\to0^+} \tanh\left(\frac{1}{x}\right)$.

As $x \to 0^+$, $\frac{1}{x} \to +\infty$,

so $\lim\limits_{x\to0^+} \tanh\left(\frac{1}{x}\right)$

$= \lim\limits_{t\to\infty} \tanh(t)$

$= \lim\limits_{t\to\infty} \frac{e^t - e^{-t}}{e^t + e^{-t}}$

$= \lim\limits_{t\to\infty} \frac{\cancel{e^t}(1 - e^{-2t})\} \to 1}{\cancel{e^t}(1 + e^{-2t})\} \to 1}$

$= \boxed{1}$

[5]

Question 3: Find the following limits if they exist:

(a) $\lim_{x\to 0} \frac{x^2}{1-\cos(x)}$ $\sim$ "$\frac{0}{0}$"

$\stackrel{H}{=} \lim_{x\to 0} \frac{2x}{\sin(x)} \sim$ "$\frac{0}{0}$"

$\stackrel{H}{=} \lim_{x\to 0} \frac{2}{\cos(x)}$

$= \boxed{2}$

[3]

(b) $\lim_{x\to 1} \frac{e^x - x}{e^x + x - 1}$ $\begin{matrix} \} \to e-1 \\ \} \to e \end{matrix}$

$= \boxed{\frac{e-1}{e}}$

[3]

(c) $\lim_{x\to 0^+} \sin(x)\ln(x)$ $\sim 0 \cdot (\infty)$

$= \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{\sin(x)}} \sim$ "$\frac{-\infty}{+\infty}$"

$\stackrel{H}{=} \lim_{x\to 0^+} \frac{\left(\frac{1}{x}\right)}{\frac{-\cos(x)}{\sin^2(x)}}$

$= \lim_{x\to 0^+} \frac{-\sin^2(x)}{x\cos(x)} \sim$ "$\frac{0}{0}$"

$\stackrel{H}{=} \lim_{x\to 0^+} \frac{-2\sin(x)\cos(x)}{\cos x - x\sin(x)}$ $\begin{matrix} \} \to 0 \\ \} \to 1 \end{matrix}$

$= \boxed{0}$

[4]

Question 4: Find the following limits if they exist:

(a) $\lim_{x\to 0^+} x^{(x^2)}$ $\sim$ "0^0"

$x^{(x^2)} = e^{x^2 \ln(x)}$ } focus on exponent !

$\lim_{x\to 0^+} x^2 \ln(x) \sim$ "$0 \cdot (-\infty)$"

$= \lim_{x\to 0^+} \frac{\ln(x)}{\left(\frac{1}{x^2}\right)} \sim$ "$\frac{-\infty}{\infty}$"

$\overset{H}{=} \lim_{x\to 0^+} \frac{\left(\frac{1}{x}\right)}{-\frac{2}{x^3}}$

$= \lim_{x\to 0^+} -\frac{x^2}{2} = 0$

So $\lim_{x\to 0^+} x^{(x^2)}$

$= e^0$

$= \boxed{1}$

[5]

(b) $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin(x)}\right)$ $\sim$ $\infty - \infty$

$= \lim_{x\to 0^+} \frac{\sin(x) - x}{x \sin(x)} \sim$ "$\frac{0}{0}$"

$\overset{H}{=} \lim_{x\to 0^+} \frac{\cos(x) - 1}{\sin(x) + x\cos(x)} \sim$ "$\frac{0}{0}$"

$\overset{H}{=} \lim_{x\to 0^+} \frac{-\sin(x)}{\cos(x) + x(-\sin(x)) + \cos(x)}$ $\begin{matrix} \} \to 0 \\ \} \to 2 \end{matrix}$

$= \boxed{0}$

[5]

Question 4: Find the following limits if they exist:

(a) Determine $f(x)$ if

$$f''(x) = 3\sqrt{x} - \frac{1}{x^2}, \quad f'(1) = 3, \quad f(1) = 0$$

$f''(x) = 3x^{1/2} - x^{-2}$

$\therefore f'(x) = \frac{3x^{3/2}}{(3/2)} - \frac{x^{-1}}{-1} + C_1 = 2x^{3/2} + \frac{1}{x} + C_1$

$f'(1) = 3 \Rightarrow 3 = 2(1)^{3/2} + \frac{1}{1} + C_1 \Rightarrow C_1 = 0$

$\therefore f'(x) = 2x^{3/2} + \frac{1}{x}$

$\therefore f(x) = 2\frac{x^{5/2}}{5/2} + \ln|x| + C_2 = \frac{4}{5}x^{5/2} + \ln|x| + C_2$

$f(1) = 0 \Rightarrow \frac{4}{5}(1)^{5/2} + \underbrace{\cancel{\ln|1|}}_{0} + C_2 = 0 \Rightarrow C_2 = \frac{-4}{5}$

$\boxed{\therefore f(x) = \frac{4}{5}x^{5/2} + \ln|x| - \frac{4}{5}}$

[5]

(b) A raindrop falls with acceleration $a(t) = 9 - 0.9t$ m/s^2. If $t = 0$ corresponds to the time at which it forms 500 m above the ground, determine how far the raindrop falls during the first 10 seconds.

$s''(t) = 9 - 0.9t$

$s'(0) = 0$

$s(0) = 500$

$\therefore s'(t) = 9t - \frac{0.9t^2}{2} + C_1$

$s'(0) = 0 \Rightarrow C_1 = 0$

$\therefore s'(t) = 9t - 0.45t^2$

$\therefore s(t) = \frac{9t^2}{2} - \frac{0.45t^3}{3} + C_2$

$s(0) = 500 \Rightarrow C_2 = 500.$

$\therefore s(t) = 4.5t^2 - 0.15t^3 + 500$

Want $|s(10) - s(0)|$

$= |4.5(10^2) - 0.15(10)^3 \cancel{+500}$
$- 0 + 0 \cancel{+500}|$

$= |450 - 150|$

$= \boxed{300 \text{ m}}$

[5]