Question 1 [10]: Determine the derivatives of the following functions. It is not necessary to simplify your final answers.

(a) $y = \frac{x^3}{3} - 3\sqrt[3]{x} + \sqrt{3} = \frac{1}{3}x^3 - 3x^{1/3} + 3^{1/2}$

$$y' = \left(\tfrac{1}{3}\right)(3)x^2 - (3)\left(\tfrac{1}{3}\right)x^{-2/3} + 0$$

or

$$= \boxed{x^2 - x^{-2/3}}$$

[3]

(b) $f(x) = (x - \cos x)\left(\frac{4}{x} - \sin x\right) = (x - \cos x)(4x^{-1} - \sin x)$

$$f'(x) = (1 + \sin x)(4x^{-1} - \sin x) + (x - \cos x)(-4x^{-2} - \cos x)$$

or

$$= \boxed{(1+\sin x)\left(\frac{4}{x} - \sin x\right) + (x - \cos x)\left(\frac{-4}{x^2} - \cos x\right)}$$

[3]

(c) $g(x) = \frac{\tan(2x)}{\pi^2 + x^2}$

$$\boxed{g'(x) = \frac{(\pi^2 + x^2)\sec^2(2x)(2) - \tan(2x)(2x)}{(\pi^2 + x^2)^2}}$$

[4]

Question 2 [10]: Determine the derivatives of the following functions. It is not necessary to simplify your final answers.

(a) $y = \sec\left(\sqrt{x} - x^5\right)$

$$\boxed{y' = \sec(x^{1/2} - x^5)\tan(x^{1/2} - x^5)\left(\tfrac{1}{2}x^{-\frac{1}{2}} - 5x^4\right)}$$

[3]

(b) $f(\theta) = \sqrt{3\theta - \theta\sin\theta} = (3\theta - \theta\sin\theta)^{1/2}$

$$\boxed{f'(\theta) = \tfrac{1}{2}(3\theta - \theta\sin\theta)^{-\frac{1}{2}}\left[3 - (\sin\theta + \theta\cos\theta)\right]}$$

[3]

(c) $g(t) = \left[t + \cos\left(\frac{1}{\sqrt{t}}\right)\right]^{121} = \left[t + \cos\left(t^{-1/2}\right)\right]^{121}$

$$\boxed{g'(t) = 121\left[t + \cos\left(t^{-\frac{1}{2}}\right)\right]^{120}\left[1 - \sin\left(t^{-\frac{1}{2}}\right)\cdot\left(\tfrac{-1}{2}\right)t^{-3/2}\right]}$$

[4]

Question 3: Determine the equation of the tangent line to the curve $x^3 - 5xy^2 + y^3 = xy - 13$ at the point $(1, 2)$.

$$\frac{d}{dx}\left[x^3 - 5xy^2 + y^3\right] = \frac{d}{dx}\left[xy - 13\right]$$

$$3x^2 - 5\left(1 \cdot y^2 + x \cdot 2yy'\right) + 3y^2 y' = 1 \cdot y + xy'$$

At $(x,y) = (1,2)$:

$$3(1)^2 - 5\left(2^2 + (1)(2)(2)y'\right) + (3)(2^2)y' = 2 + (1)y'$$

$$3 - 20 - 20y' + 12y' = 2 + y'$$

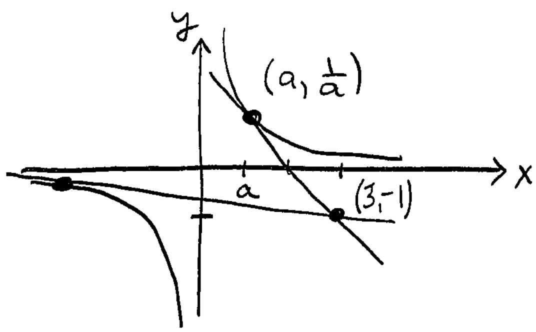
$$-9y' = 19$$

$$y' = -\frac{19}{9}$$

$\therefore$ Equation is $\boxed{y - 2 = \frac{-19}{9}(x-1)}$

[5]

Question 4: There are two tangent lines to the curve $y = 1/x$ which pass through the point $(3, -1)$. Determine the points at which these tangent lines contact the curve.



$y = \frac{1}{x}$; $y' = \frac{-1}{x^2}$

Using point of contact $(a, \frac{1}{a})$ we must have slope of tangent line

$$m = \frac{\left(\frac{1}{a} - (-1)\right)}{a-3} = \frac{\left(\frac{1}{a} + 1\right)}{a-3}.$$

But also, $m = \left.\frac{dy}{dx}\right|_{x=a} = \frac{-1}{a^2}$.

$$\therefore \frac{\left(\frac{1}{a}+1\right)}{a-3} = \frac{-1}{a^2} \Rightarrow a + a^2 = -a + 3$$

$$\Rightarrow a^2 + 2a - 3 = 0$$

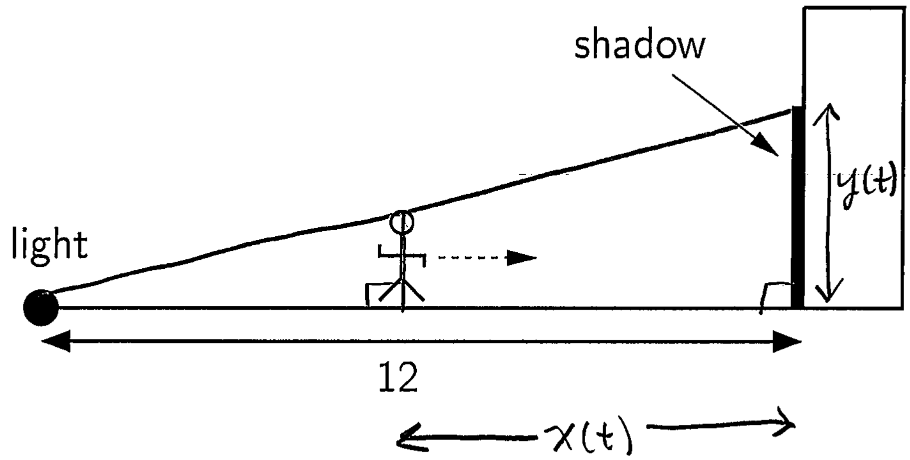
$$\Rightarrow (a-1)(a+3) = 0$$

$$\Rightarrow a = 1,\ a = -3.$$

$\therefore$ points of contact are $(1, 1)$, $(-3, \frac{-1}{3})$

[5]

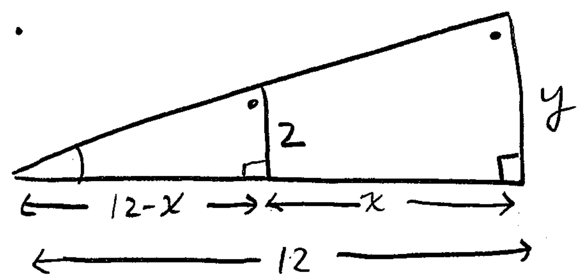
Question 5: A spotlight on the ground shines on a building wall 12 m away. If a man 2 m tall walks from the spotlight to the building at a speed of 1/2 m/s, how fast is the length of his shadow on the wall decreasing when he is 6 m from the building?



$$\frac{dx}{dt} = -\frac{1}{2}\ \frac{m}{s}.$$

Find $\frac{dy}{dt}$ when $x = 6$ m.

By similar triangles:



$$\frac{12-x}{2} = \frac{12}{y}$$

$$\therefore\ y = \frac{24}{12-x}$$

$$\therefore\ \frac{dy}{dt} = \frac{+24}{(12-x)^2}\frac{dx}{dt}$$

When $x = 6$ & $\frac{dx}{dt} = \frac{-1}{2}$,

$$\frac{dy}{dt} = \frac{24}{(12-6)^2}\left(\frac{-1}{2}\right) = -\frac{1}{3}\ \frac{m}{s}.$$

$\boxed{\therefore \text{ Shadow length is decreasing by } \frac{1}{3}\ \frac{m}{s}.}$

[10]

Question 6: Use a linear approximation (or differentials) to approximate $\sqrt[3]{27.1}$.

Here $f(x) = x^{1/3}$, $a = 27$, $f(a) = 27^{1/3} = 3$.

$f'(x) = \frac{1}{3}x^{-2/3}$; $f'(a) = \frac{1}{3}(27)^{-2/3} = \frac{1}{3}(27^{1/3})^{-2} = \frac{1}{27}$

$\therefore L(x) = f(a) + f'(a)(x-a)$

$= 3 + \frac{1}{27}(x-27)$

$\therefore (27.1)^{1/3} = f(27.1) \approx L(27.1) = 3 + \frac{1}{27}(27.1-27)$

$= 3 + \frac{1}{270}$

$= \boxed{\frac{811}{270}}$

[5]

Question 7: A sphere (ball) of radius r has volume $V = \frac{4}{3}\pi r^3$. Suppose the radius of a sphere is measured to be 10 cm with a maximum measurement error of 1/100 cm. Estimate the maximum measurement error in the calculated volume of the sphere. State units with your answer.

$V = \frac{4}{3}\pi r^3$

$dV = \frac{4}{3}\pi(3)r^2 dr = 4\pi r^2 dr$

Here $r = 10$, $dr = \frac{1}{100}$,

$\therefore dV = 4\pi \cancel{(10)^2}\left(\cancel{\frac{1}{100}}\right) = 4\pi$

$\therefore$ Maximum error in calculated volume is approximately 4π cm^3

[5]