Question 1 [10]: Evaluate the following limits if they exist. If a limit does not exist because it is $+\infty$ or $-\infty$, state which with an explanation of your reasoning.

(a) $\lim_{x\to 16} \frac{4-\sqrt{x}}{x-16}$ $\sim$ "$\frac{0}{0}$"

$$= \lim_{x\to 16} \frac{4-\sqrt{x}}{x-16} \cdot \frac{(4+\sqrt{x})}{(4+\sqrt{x})}$$

$$= \lim_{x\to 16} \frac{\cancel{(16-x)}}{-\cancel{(16-x)}(4+\sqrt{x})}$$

$$= \boxed{\frac{-1}{8}}$$

[3]

(b) $\lim_{x\to 4} \frac{x^2-5x+4}{x^2-3x-4}$ $\sim$ "$\frac{0}{0}$"

$$= \lim_{x\to 4} \frac{(x-1)\cancel{(x-4)}}{(x+1)\cancel{(x-4)}}$$

$$= \boxed{\frac{3}{5}}$$

[3]

(c) $\lim_{x\to 0} \frac{x}{\sin(x)+\tan(x)}$ $\sim$ "$\frac{0}{0}$"

$$= \lim_{x\to 0} \frac{x}{\left[\sin(x) + \frac{\sin(x)}{\cos(x)}\right]} \begin{matrix} \div x \\ \div x \end{matrix}$$

$$= \lim_{x\to 0} \frac{1}{\left[\frac{\sin(x)}{x} + \frac{\sin(x)}{x} \cdot \frac{1}{\cos(x)}\right]} = \frac{1}{1+1\cdot 1} = \boxed{\frac{1}{2}}$$

[4]

Question 2: Let

$$g(x) = \begin{cases} x^4 - cx^2 & \text{if } x < 2 \\ c^2x + 18 & \text{if } x \geq 2 \end{cases}$$

Find the constant c that makes g continuous at all real numbers.

For $x \neq 2$, g is defined by polynomials, so is continuous.
At $x = 2$, for g to be continuous we must have

$$\lim_{x\to 2^-} g(x) = \lim_{x \to 2^+} g(x) = g(2)$$

$$\Rightarrow \lim_{x\to 2^-} (x^4 - cx^2) = \lim_{x\to 2^+} (c^2x + 18) = c^2 \cdot (2) + 18$$

$$\Rightarrow 2^4 - c(2)^2 = c^2(2) + 18$$

$$\Rightarrow 2c^2 + 4c + 2 = 0$$

$$\Rightarrow 2(c+1)^2 = 0 \implies \boxed{c = -1}$$

[5]

Question 3: Determine $\lim_{x\to 0} x^3 \cos\left(\frac{x+1}{x}\right)$. The Squeeze Theorem may be helpful here. If using the theorem be sure to demonstrate that all requirements of the theorem are satisfied.

$$-|x^3| \leqslant x^3 \cos\left(\frac{x+1}{x}\right) \leqslant +|x^3|$$

$$\lim_{x\to 0} -|x^3| = 0 = \lim_{x\to 0} +|x^3|$$

$\therefore$ By the squeeze theorem, $\lim_{x\to 0} x^3 \cos\left(\frac{x+1}{x}\right) = 0$.

[5]

Question 4 [10]: Evaluate the following limits if they exist. If a limit does not exist because it is $+\infty$ or $-\infty$, state which with an explanation of your reasoning.

(a) $\lim_{x \to -6^-} \frac{2x+12}{|x+6|}$ $\sim$ "$\frac{0}{0}$"

$$= \lim_{x \to -6^-} \frac{2\cancel{(x+6)}}{-\cancel{(x+6)}}$$

$$= \boxed{-2}$$

[5]

(b) $\lim_{x\to\infty} (x - \sqrt{x})$ $\sim$ "$\infty - \infty$"

$$= \lim_{x\to\infty} \underbrace{\sqrt{x}}_{\to\infty} \underbrace{(\sqrt{x} - 1)}_{\to\infty}$$

$$= \boxed{\infty}$$

[5]

Question 5 [10]:

(a) Use the limit definition of the derivative to find $f'(x)$ if $f(x) = \frac{x}{x+1}$. Neatly show all steps and use proper notation. (No credit will be given if $f'(x)$ is found using derivative rules.)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{x+h}{x+h+1} - \frac{x}{x+1}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{(x+h)(x+1) - (x+h+1)(x)}{(x+h+1)(x+1)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\cancel{x^2} + \cancel{hx} + \cancel{x} + h \cancel{- x^2} \cancel{- hx} \cancel{- x}}{(x+h+1)(x+1)}\right]$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}}\left[\frac{\cancel{h}}{(x+h+1)(x+1)}\right]$$

$$= \boxed{\frac{1}{(x+1)^2}}$$

[8]

(b) Give a one sentence statement describing the relationship between $f'(2)$ and the graph of $y = \frac{x}{x+1}$.

$f'(2)$ is the slope of the tangent line to the graph of $y = \frac{x}{x+1}$ at the point where $x = 2$.

[2]

Question 6: Find y' if $y = \dfrac{\sin(x)}{2} - \dfrac{2}{\sqrt{x}}$ $= \frac{1}{2}\sin(x) - 2x^{-\frac{1}{2}}$

$$y' = \frac{1}{2}\cos(x) - 2\left(-\frac{1}{2}\right)x^{-3/2}$$

$$= \boxed{\frac{1}{2}\cos(x) + x^{-3/2}}$$

[3]

Question 7: Find an equation of the tangent line to $y = 3x^2 - x^3$ at the point where $x = 1$.

- At $x=1$, $y = 3(1)^2 - (1)^3 = 2$, so $(1,2)$ is a point on the tangent line.
- Slope of the tangent line is $\left.\frac{dy}{dx}\right|_{x=1} = \left. 6x - 3x^2 \right|_{x=1} = 3$

$\therefore$ Equation of tangent line is

$$\boxed{y - 2 = 3(x-1)}$$

[4]

Question 8: Suppose a particle moves along a straight line according to the position function

$$s(t) = t^3 - 9t^2 + 15t + 10$$

When, if ever, is the particle's acceleration zero?

Solve $s''(t) = 0$ for t:

$s'(t) = 3t^2 - 18t + 15$

$s''(t) = 6t - 18$

$s''(t) = 0$ at $\boxed{t=3}$

[3]