Question 1 [12 points]: Evaluate the following limits, if they exist. If a limit does not exist because it is $\pm\infty$, state which it is and include an explanation of your reasoning. You may use the fact that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

(a) $\lim_{x\to -5} \frac{x^2+3x-10}{x^2+8x+15}$ $= \lim_{x\to -5} \frac{\cancel{(x+5)}(x-2)}{\cancel{(x+5)}(x+3)} = \frac{-7}{-2} = \boxed{\frac{7}{2}}$

(b) $\lim_{x\to 9} \frac{4-\sqrt{x+7}}{9-x}$ $\cdot \frac{4+\sqrt{x+7}}{4+\sqrt{x+7}} = \lim_{x\to 9} \frac{16-(x+7)}{(9-x)(4+\sqrt{x+7})}$

$= \lim_{x\to 9} \frac{\cancel{(9-x)}}{\cancel{(9-x)}(4+\sqrt{x+7})}$

$= \boxed{\frac{1}{8}}$

(c) $\lim_{x\to 0} \frac{\tan(2x)+3x}{\sin(2x)}$ $= \lim_{x\to 0} \frac{\left(\frac{\sin(2x)}{\cos(2x)} + 3x\right)}{\sin(2x)}$ $\begin{matrix} \div 2x \\ \div 2x \end{matrix}$

$= \lim_{x\to 0} \frac{\frac{\sin(2x)}{2x}\frac{1}{\cos(2x)} + \frac{3x}{2x}}{\frac{\sin(2x)}{2x}}$

$= \frac{1\cdot 1 + \frac{3}{2}}{1} = \boxed{\frac{5}{2}}$

(d) $\lim_{x\to 2^-} \frac{-4}{x^2-x-2}$

$= \lim_{x\to 2^-} \frac{-4}{\underbrace{(x-2)}_{\to 0^-}\underbrace{(x+1)}_{\to 3^-}} = \boxed{+\infty}$

Question 2 [8 points]: Consider the function $f(x) = x^3 - 2x^2 - 1$.

(a) Find an equation of the tangent line to the graph of $f(x)$ at $x = -2$.

$f(x) = x^3 - 2x^2 - 1 \quad ; \quad f(-2) = (-2)^3 - 2(-2)^2 - 1 = -17$

$f'(x) = 3x^2 - 4x \quad ; \quad f'(-2) = 3(-2)^2 - 4(-2) = 20$

$\therefore \quad y - (-17) = 20(x - (-2))$

$\boxed{y + 17 = 20(x+2)}$

or $\boxed{y = 20x + 23}$

(b) Find the x-value(s) where the tangent line to the graph of $f(x)$ is parallel to the line $y = 4x - 2$.

Solve $f'(x) = 4$

$3x^2 - 4x = 4$

$3x^2 - 4x - 4 = 0$

$3x^2 - 6x + 2x - 4 = 0$

$3x(x-2) + 2(x-2) = 0$

$(3x+2)(x-2) = 0$

$3x + 2 = 0 \quad , \quad x - 2 = 0$

$\boxed{x = \frac{-2}{3} \quad , \quad x = 2}$

Question 3 [15 points]: Differentiate the following functions (you do not have to simplify your answers):

(a) $y = 4\ln(3x^4 - 5x^2)$

$$y' = \frac{4}{3x^4-5x^2} \cdot (12x^3 - 10x)$$

(b) $y = \dfrac{e^x - x}{x^2 + \sin x}$

$$y' = \frac{(x^2+\sin x)(e^x-1) - (e^x - x)(2x+\cos x)}{(x^2+\sin x)^2}$$

(c) $y = 5^x \sec x$

$$y' = (5^x \ln 5)\sec x + 5^x(\sec x \tan x)$$

(d) $y = \left(5x^3 - \dfrac{7}{x^2}\right) e^{\cos x}$ $= (5x^3 - 7x^{-2})e^{\cos x}$

$$y' = (15x^2 + 14x^{-3})e^{\cos x} + (5x^3 - 7x^{-2})e^{\cos x}(-\sin x)$$

(e) $y = \tan\left(\sqrt{x^3 - \log_5 x}\right)$

$$y' = \sec^2\left(\sqrt{x^3 - \log_5 x}\right) \cdot \frac{1}{2}(x^3 - \log_5 x)^{-\frac{1}{2}} \cdot \left(3x^2 - \frac{1}{x\ln 5}\right)$$

Question 4 [**5 points**]: Use the definition of the derivative to find $f'(x)$ for $f(x) = \frac{3}{x+7}$. (A score of 0 will be given if $f'(x)$ is found using the differentiation rules.)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{3}{x+h+7} - \frac{3}{x+7} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{\cancel{3x} + \cancel{21} - \cancel{3x} - 3h - \cancel{21}}{(x+h+7)(x+7)} \right]$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}} \cdot \frac{-3\cancel{h}}{(x+h+7)(x+7)}$$

$$= \frac{-3}{(x+7)^2}$$

Question 5 [10 points]:

(a) Use implicit differentiation to find $\frac{dy}{dx}$ for $e^{xy^2} = 4x^2 - \cos y$.

$$\frac{d}{dx}\left[e^{xy^2}\right] = \frac{d}{dx}\left[4x^2 - \cos y\right]$$

$$e^{xy^2} \cdot \left[y^2 + 2xyy'\right] = 8x + \sin y \cdot y'$$

$$y^2 e^{xy^2} + 2xy e^{xy^2} y' = 8x + \sin y \cdot y'$$

$$y'\left[2xy e^{xy^2} - \sin y\right] = 8x - y^2 e^{xy^2}$$

$$y' = \frac{8x - y^2 e^{xy^2}}{2xy e^{xy^2} - \sin y}$$

(b) Use logarithmic differentiation to find $\frac{dy}{dx}$ for $y = \frac{\sin^{10} x}{7^{(x^2-4x+1)}}$.

$$\ln y = \ln\left[\frac{(\sin x)^{10}}{7^{x^2-4x+1}}\right]$$

$$\ln y = 10 \ln(\sin x) - (x^2 - 4x + 1) \ln 7$$

$$\frac{1}{y} y' = \frac{10}{\sin x} \cdot \cos x - (2x-4) \ln 7$$

$$\therefore y' = \frac{\sin^{10} x}{7^{x^2-4x+1}}\left[\frac{10 \cos x}{\sin x} - (2x-4)\ln 7\right]$$

or $$y' = \frac{\sin^{10} x}{7^{x^2-4x+1}}\left[10 \cot x - \ln(7^{2x-4})\right].$$

Question 6 [10 points]:

(a) Find the general antiderivative for each of the following functions:

(i) $f(x) = -\sec^2 x - 4\sin x + 7e^x$

$$\boxed{F(x) = -\tan x + 4\cos x + 7e^x + C}$$

(ii) $f(x) = \dfrac{3x^6 - 4x + 2\sqrt[4]{x^3}}{x^2} = 3x^4 - 4\left(\frac{1}{x}\right) + 2x^{-\frac{5}{4}}$

$$F(x) = \frac{3x^5}{5} - 4\ln|x| + 2\,\frac{x^{-\frac{1}{4}}}{\left(-\frac{1}{4}\right)} + C$$

$$= \boxed{\frac{3}{5}x^5 - 4\ln|x| - 8x^{-\frac{1}{4}} + C}$$

(b) Find the function $g(t)$ such that $g'(t) = 5t^4 + 9t^2$ and $g(-1) = 7$.

$$g(t) = \frac{5t^5}{5} + \frac{9t^3}{3} + C$$

$$= t^5 + 3t^3 + C.$$

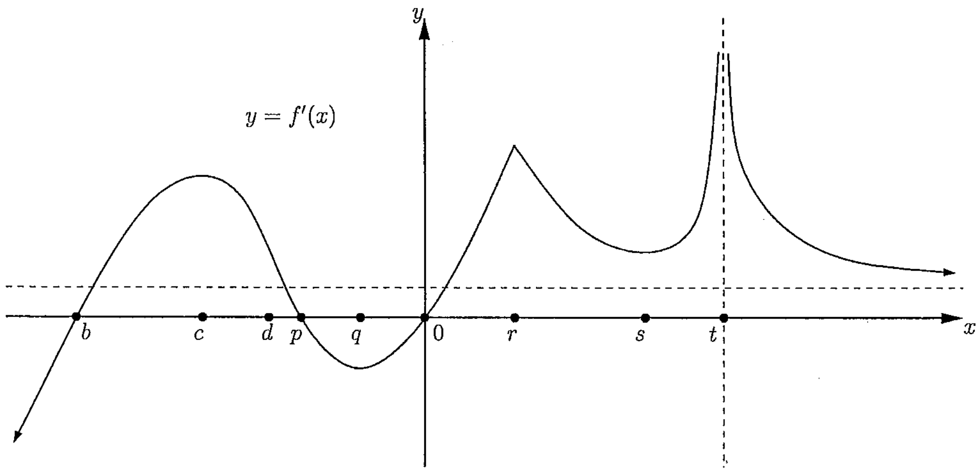
$$g(-1) = 7 \Rightarrow 7 = (-1)^5 + 3(-1)^3 + C$$

$$7 = -1 - 3 + C$$

$$C = 11$$

$$\therefore \boxed{g(t) = t^5 + 3t^3 + 11}$$

Question 7 [**10 points**]: The graph of $f'(x)$ is shown below (note this is the graph of $f'(x)$, not $f(x)$):



(a) On what interval(s) is f decreasing?

$f'(x) < 0 \Rightarrow f$ decreasing,

$\therefore \ (-\infty, b) \cup (p, 0)$

(b) At what x-value(s) does $f(x)$ have local minima?

f' changing from negative to positive $\Rightarrow$ local min.

$\therefore \ x = b, \ x = 0$

(c) On what interval(s) is the graph of $f(x)$ concave up?

f' increasing $\Rightarrow f'' > 0 \Rightarrow f$ concave up.

$\therefore (-\infty, c) \cup (q, r) \cup (s, t)$

(d) At what x-value(s) does the graph of $f(x)$ have inflection points?

f' changing from increasing to decreasing (or vice versa)
$\Rightarrow$ inflection point

$\therefore \ x = c, \ x = q, \ x = r, \ x = s$

(e) At what x-value(s) does $f''(x)$ not exist?

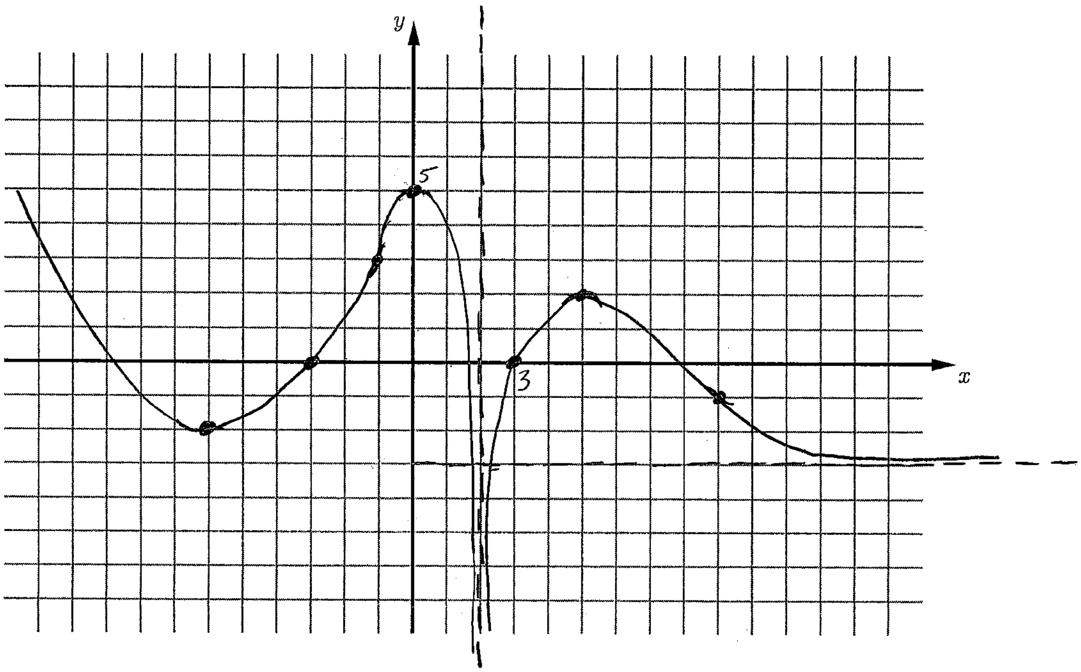
$x = r, \ x = t.$

Question 8 [5 points]: You have completed the analysis of a function $f(x)$ and found the information listed below. Sketch the graph of $y = f(x)$.

- The domain of $f(x)$ is $(-\infty, 2), (2, \infty)$.
- $f(x)$ has the following function values:

x	-6	-3	-1	0	3	5	9
$f(x)$	-2	0	3	5	0	2	-1

- $\lim_{x\to\infty} f(x) = -3$, $\lim_{x\to-\infty} f(x) = \infty$
- $\lim_{x\to 2} f(x) = -\infty$
- $f'(-6) = f'(0) = f'(5) = 0$
- $f'(x) > 0$ on $(-6, 0)$ and $(2, 5)$
- $f'(x) < 0$ on $(-\infty, -6)$, $(0, 2)$ and $(5, \infty)$
- $f''(-1) = f''(9) = 0$
- $f''(x) > 0$ on $(-\infty, -1)$ and $(9, \infty)$
- $f''(x) < 0$ on $(-1, 2)$ and $(2, 9)$



Question 9 [12 points]: The function $f(x) = \frac{x}{x^2+1}$ has $f'(x) = \frac{1-x^2}{(x^2+1)^2}$ and $f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}$.

(a) Find the intervals on which $f(x)$ is increasing or decreasing.

$f'(x) = \frac{1-x^2}{(x^2+1)^2} = 0$ at $x = 1, -1$

	-2	-1	0	1	2
$f'(x) = \frac{1-x^2}{(x^2+1)^2}$:	$-$	0	$+$	0	$-$
$f(x) = \frac{x}{x^2+1}$:	$\searrow$	$-\frac{1}{2}$	$\nearrow$	$\frac{1}{2}$	$\searrow$

$\therefore$ f is increasing on $(-1, 1)$, decreasing on $(-\infty, -1) \cup (1, \infty)$.

(b) Find the local maximum and minimum values of $f(x)$.

f has a local minimum of $-\frac{1}{2}$ at $x = -1$.

f has a local maximum of $\frac{1}{2}$ at $x = 1$.

(c) Find the intervals on which $f(x)$ is concave up or concave down.

$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3} = 0$ at $x = 0, \sqrt{3}, -\sqrt{3}$

	-2	$-\sqrt{3}$	-1	0	1	$\sqrt{3}$	2
$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}$:	$-$	0	$+$	0	$-$	0	$+$
$f(x) = \frac{x}{x^2+1}$:	$\frown$	$-\frac{\sqrt{3}}{4}$	$\smile$	0	$\frown$	$\frac{\sqrt{3}}{4}$	$\smile$

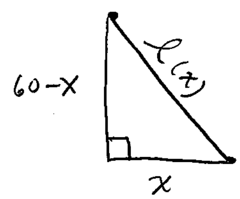
$\therefore$ f is concave up on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$;

concave down on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$.

(d) Find the inflection points of $f(x)$.

$\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), (0, 0), \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$.

Question 10 [8 points]: A straight wire is 60 cm long. The wire is bent into the shape of an L, where the bend forms a right angle. What is the shortest possible distance between the two ends of the bent wire?



$$\ell(x) = \left[x^2 + (60-x)^2\right]^{1/2}, \qquad 0 \le x \le 60$$

$$\ell'(x) = \frac{1}{2}\left[x^2 + (60-x)^2\right]^{-\frac{1}{2}} \cdot \left[2x + 2(60-x)(-1)\right]$$

$$= \frac{4x - 120}{2\sqrt{x^2 + (60-x)^2}}$$

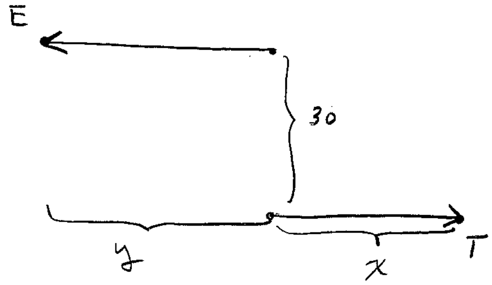
$$= \frac{2x - 60}{\sqrt{x^2 + (60-x)^2}}$$

- $\ell'(x) = 0$ at $x = 30$
- $\ell'(x)$ not exist? no such x.

x	$\ell(x) = \sqrt{x^2 + (60-x)^2}$
0	60
30	$\sqrt{30^2 + 30^2} = 30\sqrt{2}$
60	60

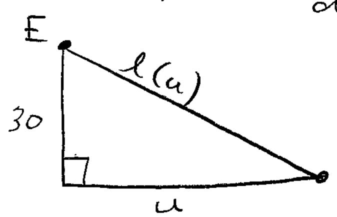
$\therefore$ The shortest possible distance is $30\sqrt{2}$ cm.

Question 11 [10 points]: Two ships, the Erebus and the Terror, are sailing the high seas in search of the Northwest Passage. At 6:00 am, the Erebus is 30 km north of the Terror. The Erebus is sailing west at 8 km/hr and the Terror is sailing east at 12 km/hr. How fast is the distance between the ships changing at 8:00 am?



$\frac{dy}{dt} = 8 \frac{km}{h}$. $\frac{dx}{dt} = 12 \frac{km}{h}$

Let $u = x + y$, so $\frac{du}{dt} = 20 \frac{km}{h}$:



$\frac{du}{dt} = 20$

Find $\frac{dl}{dt}$ when $u = (2)(8) + (2)(12) = 40$.

$l = [30^2 + u^2]^{1/2}$.

$\frac{dl}{dt} = \frac{1}{2}[30^2 + u^2]^{-\frac{1}{2}}\left[2u \frac{du}{dt}\right]$

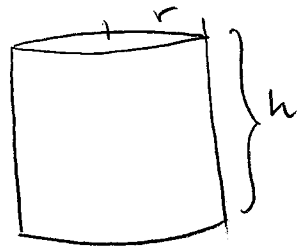
when $u = 40$:

$\frac{dl}{dt} = \frac{1}{\cancel{2}}[30^2 + 40^2]^{-\frac{1}{2}}[\cancel{2} \cdot 40 \cdot 20]$

$= \frac{800}{50} = 16 \frac{km}{h}$.

$\therefore$ Distance between ships is increasing by $16 \frac{km}{h}$.

Question 12 [10 points]: A cylindrical can with a bottom but no top is to be made from 300π square meters of aluminum. Find the largest possible volume of such a can. (Note that the volume of cylinder is $V = \pi r^2 h$.)



$$V = \pi r^2 h$$

$$S = \pi r^2 + 2\pi r h = 300\pi$$

Maximize $V = \pi r^2 h$

subject to $\pi r^2 + 2\pi r h = 300\pi$

$$\pi r^2 + 2\pi r h = 300\pi$$

$$h = \frac{300\cancel{\pi} - \cancel{\pi} r^2}{2\cancel{\pi} r} = \frac{300 - r^2}{2r}$$

$$\therefore \quad V(r) = \pi r^2 \left[\frac{300 - r^2}{2r}\right] = \frac{\pi}{2} r (300 - r^2)$$

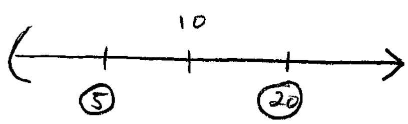
$$V'(r) = \frac{\pi}{2}\left[300 - r^2 + r(-2r)\right]$$

$$= \frac{\pi}{2}\left[300 - 3r^2\right]$$

$$V'(r) = 0 \Rightarrow 300 - 3r^2 = 0$$

$$\therefore \quad r = \pm\sqrt{\frac{300}{3}} = 10, \cancel{-10}$$

since $r > 0$.



$V'(r) = \frac{\pi}{2}[300 - 3r^2]$: $+$ $\quad 0 \quad$ $-$

$V(r) = \frac{\pi}{2} r(300 - r^2)$: $\nearrow$ $\quad 1000\pi \quad$ $\searrow$

$\therefore$ The largest possible volume is $1000\pi \ m^3$.

Question 13 [5 points]: Consider the function $f(x) = ax^3 + bx^2 + c$. Find the values of a, b and c so that $(-1, 0)$ is a point on $f(x)$ and so that $f(x)$ has a point of inflection at $(1, 1)$.

$$f(-1) = 0 \Rightarrow -a + b + c = 0 . \quad (1)$$

$$f(1) = 1 \Rightarrow a + b + c = 1 \quad (2)$$

$$f''(1) = 0 \Rightarrow \frac{d}{dx}\left[3ax^2 + 2bx\right]\Big|_{x=1} = 0$$

$$\Rightarrow 6ax + 2b \Big|_{x=1} = 0$$

$$6a + 2b = 0 . \quad (3)$$

$(1) - (2)$: $-2a = -1$

$\therefore \boxed{a = \frac{1}{2}}$

using (3) : $6\left(\frac{1}{2}\right) + 2b = 0$

$\boxed{b = -\frac{3}{2}}$

using (2): $\frac{1}{2} - \frac{3}{2} + c = 1$

$c = 1 + \frac{3}{2} - \frac{1}{2}$

$\boxed{c = 2}$