Second Derivatives and Shapes of Curves

We have seen how $f'(x)$ gives us information about how a function increases and decreases, in particular:

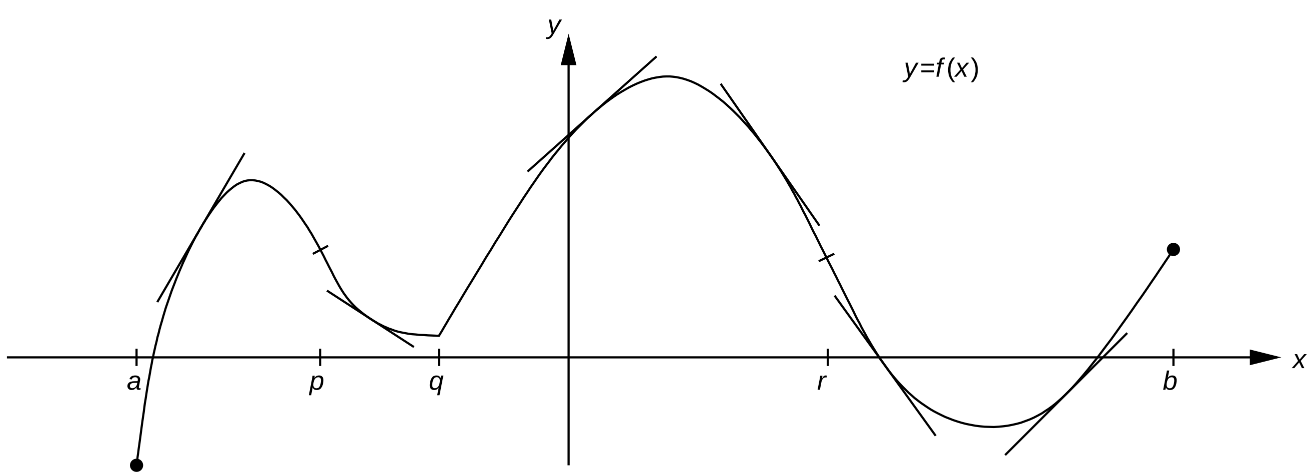
(i) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(ii) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Now we examine what the second derivative $f''(x)$ tells us about f and its graph. First, recall

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left[f'(x)\right] \\ &= \text{the rate of change of } f'(x) \\ &= \text{the slope of tangent lines to } y = f'(x) \end{aligned}$$

To see how this applies to the graph of $y = f(x)$, consider a general graph on which some tangent lines have been drawn:



Reading the graph left to right (as always!), notice:

(i) Slopes of tangent lines decrease over the intervals (a, p) and (q, r). Over these intervals tangent lines lie above the graph, and the graph itself bends downward.

(ii) Slopes of tangent lines increase over the intervals (p, q) and (r, b). Over these intervals tangent lines lie below the graph, and the graph itself bends upward.

(iii) The transitions between increasing and decreasing tangent slopes, that is, transitions between bending trends, occur at $x = p$, $x = q$ and at $x = r$.

Now, extend what we learned about first derivatives to second derivatives:

$f''(x) > 0 \Rightarrow f'(x)$ is increasing $\Rightarrow$ tangent slopes are increasing $\Rightarrow$ graph of $y = f(x)$ bends upward
$f''(x) < 0 \Rightarrow f'(x)$ is decreasing $\Rightarrow$ tangent slopes are decreasing $\Rightarrow$ graph of $y = f(x)$ bends downward

The manner in which a graph bends or curves is known as **concavity**, and this property is described by the second derivative.

Definitions

concave up: if the graph of f lies above all of its tangents on an interval, then the graph is said to be **concave up** on the interval. Think: the graph bends upwards.

concave down: if the graph of f lies below all of its tangents on an interval, then the graph is said to be **concave down** on the interval. Think: the graph bends downwards.

inflection point: an inflection point on the curve $y = f(x)$ is a point $(c, f(c))$ at which

(i) f is continuous, and

(ii) the graph of $y = f(x)$ changes concavity (i.e. changes from concave up to concave down or vice versa.)

So, referring to the graph above, we would say:

- f is concave down on (a, p) and (q, r);
- f is concave up on (p, q) and (r, b);
- f has inflection points at $(p, f(p))$, $(q, f(q))$ and $(r, f(r))$.

Concavity Test

Concavity as described by the second derivative is formalized in the **Concavity Test:**

(i) If $f''(x) > 0$ on an interval, then the graph of $y = f(x)$ is concave up on the interval.

(ii) If $f''(x) < 0$ on an interval, then the graph of $y = f(x)$ is concave down on the interval.

Observe on our graph: whenever the graph of $y = f(x)$ changes from concave up to concave down, or vice versa, $f'(x)$ changes from increasing to decreasing, or vice versa. That is, $f''(x)$ changes from positive to negative, or vice versa. This may occur at points where $f''(x) = 0$ or $f''(x)$ does not exist, or at points where the original function $f(x)$ is not defined. Putting this all together:

To determine the intervals of concavity of a function f:

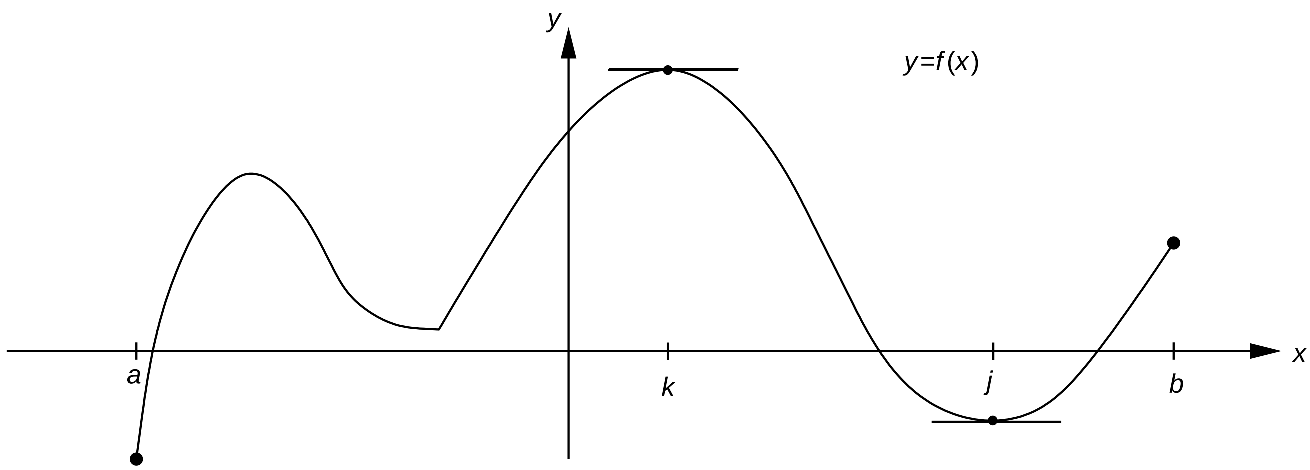
(i) Find points at which f'' changes sign (from positive to negative or vice versa). f'' can change sign at points where

- $f''(x) = 0$
- $f''(x)$ does not exist
- $f(x)$ is not defined

(ii) Test $f''(x)$ on the subintervals defined by the points from (i).

The Second Derivative Test

The second derivative can also be used to easily identify when a critical number corresponds to a relative minimum or maximum, so provides an alternative to the first derivative test. Consider the relative maximum at $x = k$ and the relative minimum at $x = j$ shown on the following graph and consider f' and f'' at these two points:



The Second Derivative Test: Suppose $f''(x)$ is continuous near $x = c$.

(i) If $f'(c) = 0$ and $f''(c) > 0$ then f has a relative minimum at $x = c$.

(ii) If $f'(c) = 0$ and $f''(c) < 0$ then f has a relative maximum at $x = c$.