Question 1 [10]: Determine the derivatives of the following functions. It is not necessary to simplify your final answers.

(a) $y = \ln(\sec x)$

$$\boxed{y' = \frac{1}{\sec(x)} \cdot \sec(x)\tan(x)}$$

[3]

(b) $f(x) = \sin^{-1}(e^x)$

$$\boxed{f'(x) = \frac{1}{\sqrt{1-(e^x)^2}} \cdot e^x}$$

[3]

(c) $y = 10^{\arctan(\pi x)}$

$$\boxed{y' = 10^{\arctan(\pi x)} \cdot \ln(10) \cdot \frac{1}{1+(\pi x)^2} \cdot \pi}$$

[4]

(d) $g(x) = \ln(e^{-x} + xe^{-x}) = \ln\left[e^{-x}(1+x)\right] = \ln(e^{-x}) + \ln(1+x)$

$\therefore g(x) = -x + \ln(1+x)$

$$\boxed{f'(x) = -1 + \frac{1}{1+x}}$$

[4]

Question 2 [10]:

(a) Solve for x: $\ln(x+1) + \ln(x-1) = 1$

$$\ln(x+1)(x-1) = 1$$
$$(x+1)(x-1) = e$$
$$x^2 - 1 = e$$
$$x^2 = e+1$$
$$x = \sqrt{e+1}, \boxed{\cancel{-\sqrt{e+1}}}$$

not a solution ($\ln(x+1)$, $\ln(x-1)$ not defined at $x = -\sqrt{e+1}$.)

$$\boxed{\therefore x = \sqrt{e+1}}$$

[3]

(b) Find the exact value of $\tan(\arcsin(-1/2))$.

$\arcsin\left(\frac{-1}{2}\right)$ = angle θ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin\theta = -\frac{1}{2}$

$$= -\frac{\pi}{6}.$$

$$\therefore \tan\left(\arcsin\left(\frac{-1}{2}\right)\right) = \tan\left(\frac{-\pi}{6}\right) = \frac{\sin(-\pi/6)}{\cos(-\pi/6)} = \frac{(-1/2)}{\left(\sqrt{3}/2\right)} = \boxed{\frac{-1}{\sqrt{3}}}$$

[3]

Question 3: Use logarithmic differentiation to find y' where $y = \dfrac{\sqrt{x+1}\,(x+5)^3}{(x+3)^5}$.

$$\ln(y) = \ln\sqrt{x+1} + \ln(x+5)^3 - \ln(x+3)^5$$
$$= \frac{1}{2}\ln(x+1) + 3\ln(x+5) - 5\ln(x+3)$$

$$\therefore \frac{1}{y}y' = \frac{1}{2(x+1)} + \frac{3}{x+5} - \frac{5}{x+3}$$

$$\therefore \boxed{y' = \left[\frac{1}{2(x+1)} + \frac{3}{x+5} - \frac{5}{x+3}\right]\frac{\sqrt{x+1}\,(x+5)^3}{(x+3)^5}}$$

[4]

Question 4: Determine the following limits:

(a) $\lim_{x\to 0} \frac{\sin(5x)}{\tan(3x)}$ $\sim$ "$\frac{0}{0}$"

$\overset{H}{=} \lim_{x\to 0} \frac{5\cos(5x)}{3\sec^2(3x)}$

$= \boxed{\frac{5}{3}}$

[3]

(b) $\lim_{x\to 0^+} \sqrt{x}\ln(x)$ $\sim$ "$0\cdot(-\infty)$"

$= \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1/2}}$

$\overset{H}{=} \lim_{x\to 0^+} \frac{(1/x)}{(-\frac{1}{2}x^{-3/2})}$

$= \lim_{x\to 0^+} -2x^{1/2}$

$= \boxed{0}$

[3]

(c) $\lim_{x\to 1^+} x^{1/(1-x)}$ $\sim$ "$1^{-\infty}$"

$= \lim_{x\to 1^+} e^{\left(\frac{1}{1-x}\right)\ln(x)}$

$= \lim_{x\to 1^+} e^{\frac{\ln(x)}{1-x}}$

for $\lim_{x\to 1^+} \frac{\ln(x)}{1-x}$ $\sim$ "$\frac{0}{0}$"

$\overset{H}{=} \lim_{x\to 1^+} \frac{1/x}{-1}$

$= -1$

$= \boxed{e^{-1}}$

[4]

Question 5: Determine the absolute minimum and maximum values of $f(x) = x^2 e^{-x}$ on the interval $[-1, 3]$. (Note: it may be useful to know that e^2 is approximately 7, and that e^3 is approximately 20.)

$f(x) = x^2 e^{-x}$

$f'(x) = 2xe^{-x} - x^2 e^{-x} = xe^{-x}(2-x)$

• $f'(x) = 0$? $x = 0, 2$

• $f'(x)$ not exist? no such x.

x	$f(x) = x^2 e^{-x}$
-1	$e \approx 2.7 \leftarrow$ abs. max.
0	$0 \leftarrow$ abs. min
2	$\frac{4}{e^2} \approx \frac{4}{7}$
3	$\frac{9}{e^3} \approx \frac{9}{20}$

end pts. (-1, 3); C.N. (0, 2)

$\therefore$ f has an abs. max. of e at $x = -1$, an abs. min. of 0 at $x = 0$.

[6]

Question 6: For this question use $f(x) = \dfrac{x}{x^2+1}$

(a) Determine the intervals on which f is increasing or decreasing

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{x^2+1} = \frac{1-x^2}{1+x^2} = \frac{(1-x)(1+x)}{\underbrace{1+x^2}_{\neq 0}}$$

- $\underline{f'(x)=0}$? $x = 1, -1$
- $f'(x)$ $\underline{\text{not exist}}$? no such x

($\underline{\text{Note}}$: $x^2+1 \neq 0$ for all real x, so f is continuous and differentiable on $(-\infty, \infty)$.)

Crit. nums.:		-1		1	
test. pts.:	-2		0		2
$f'(x) = \frac{1-x^2}{1+x^2}$:	$-$	0	$+$	0	$-$
$f(x) = \frac{x}{1+x^2}$:	$\searrow$	$\frac{-1}{2}$	$\nearrow$	$\frac{1}{2}$	$\searrow$

$\therefore$ f is increasing on $(-1, 1)$;
f is decreasing on $(-\infty, -1) \cup (1, \infty)$.

[8]

(b) Determine the local (or relative) maximum and minimum values of f.

f has a local maximum of $\frac{1}{2}$ at $x=1$;
f has a local minimum of $\frac{-1}{2}$ at $x=-1$.

[2]