Question 1: Suppose $f(z) = u(x,y) + iv(x,y)$ is entire where $u(x,y) = xy + x + 2y - 5$ and $f(0) = -5 + 2i$. Determine $f(1)$.

Find $v(x,y)$: $u_x = v_y \Rightarrow v_y = y+1$

$\Rightarrow v = \frac{y^2}{2} + y + g(x)$

$u_y = -v_x \Rightarrow x + 2 = -g'(x)$

$\Rightarrow g(x) = \frac{-x^2}{2} - 2x + k$

$\therefore f(z) = \left[xy + x + 2y - 5\right] + i\left[\frac{y^2}{2} + y - \frac{x^2}{2} - 2x + k\right]$

$f(0) = -5 + 2i \Rightarrow -5 + ik = -5 + 2i$

$\Rightarrow k = 2$

$\therefore f(1) = f(1 + 0i) = \left[\cancel{1\cdot 0} + 1 + \cancel{2\cdot 0} - 5\right] + i\left[\cancel{\frac{0^2}{2}} + \cancel{0} - \frac{1^2}{2} \cancel{- 2\cdot 1 + 2}\right]$

$= \boxed{-4 - \frac{1}{2}i}$

[7]

Question 2: Suppose $f(z) = u(x,y) + iv(x,y)$ is analytic in some domain D and that $u(x,y) > 0$ on D. Explain why Log|f(z)| is harmonic on D. [note: here $|f(z)|$ is real, so Log|f(z)| = ln |f(z)|.]

Since $u(x,y) > 0$, $f(z)$ is never on the branch cut of $\text{Log}(z)$, so $\text{Log}(f(z))$ is analytic on D.

$\therefore \text{Re}(\text{Log}(f(z))) = \text{Log}|f(z)|$ is harmonic on D.

[3]

Question 3: Simplify

$$\sin(iz - \frac{\pi}{2}) + \cos(iz)$$

$$= \frac{1}{2i}\left[e^{i(iz - \pi/2)} - e^{-i(iz - \frac{\pi}{2})}\right] + \frac{1}{2}\left[e^{i(iz)} + e^{-i(iz)}\right]$$

$$= \frac{1}{2i}\left[e^{-z}\big(\overset{-i}{\cancel{e^{-i\pi/2}}}\big) - e^{z}\big(\underset{i}{\cancel{e^{i\pi/2}}}\big)\right] + \frac{1}{2}\left[e^{-z} + e^{z}\right]$$

$$= \frac{1}{2\cancel{i}}\left[e^{-z}(-\cancel{i}) - e^{z}(\cancel{i})\right] + \frac{1}{2}\left[e^{-z} + e^{z}\right]$$

$$= \frac{-e^{-z} - e^{z} + e^{-z} + e^{z}}{2}$$

$$= \boxed{0}$$

[5]

Question 4: Determine the largest set on which $f(z) = \text{Log}(4 + \text{i} - \text{z})$ is analytic and compute $f'(z)$.

f fails to be analytic iff

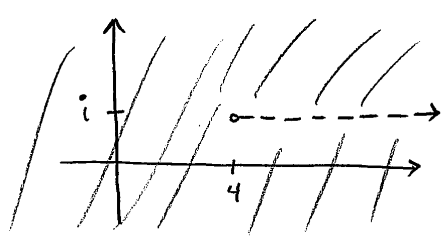
$$4 + i - z = -x + i0 \quad \text{where } x \geq 0$$

$$\iff \quad z = (4 + x) + i, \quad x \geq 0$$

$$\iff \quad z = t + i, \quad t \geq 4$$

$\therefore$ f is analytic on $\mathbb{C} \setminus \{t + i \mid t \geq 4\}$;

i.e. on



$$f'(z) = \frac{-1}{4 + i - z}.$$

[5]

Question 5: Find all solutions to

$$\mathrm{Log}(z^2-1)=\frac{i\pi}{2}$$

$$e^{\mathrm{Log}(z^2-1)} = e^{i\pi/2} = i$$

$$z^2-1 = i$$

$$z^2 = 1+i$$

$$z = (1+i)^{1/2}$$

$$= \left(\sqrt{2}\, e^{i\pi/4}\right)^{1/2}$$

$$= 2^{1/4} e^{i\left(\frac{\pi}{4}+2k\pi\right)\frac{1}{2}}, \quad k=0,1$$

$$= \boxed{2^{\frac{1}{4}} e^{i\pi/8}, \quad 2^{1/4} e^{i9\pi/8}.}$$

[5]

Question 6: Compute both $(i-1)^{2i}$ and $[(i-1)^2]^i$ using the principal branch of the logarithm. (Your answers should not be the same. This problem shows that for complex z, α and β, it is not true in general that $(z^\alpha)^\beta = z^{\alpha\beta}$.)

$$i-1 = \sqrt{2}\, e^{i\,3\pi/4}$$

$$(i-1)^{2i} = e^{2i\,\mathrm{Log}\left[\sqrt{2}\, e^{i\,3\pi/4}\right]}$$

$$= e^{2i\left[\mathrm{Log}\sqrt{2} + i\,3\pi/4\right]}$$

$$= e^{-3\pi/2 + i\,2\,\mathrm{Log}(\sqrt{2})}$$

$$= \boxed{e^{-3\pi/2 + i\,\mathrm{Log}\,2}}$$

$$\left[(i-1)^2\right]^i = \left[\left(\sqrt{2}\, e^{i3\pi/4}\right)^2\right]^i$$

$$= \left[2\, e^{i\,3\pi/2}\right]^i$$

NOTE!

$$= \left[2\, e^{-i\frac{\pi}{2}}\right]^i$$

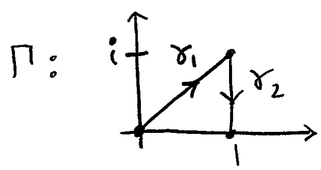
$$= e^{i\,\mathrm{Log}\left[2e^{-i\pi/2}\right]}$$

$$= e^{i\left[\mathrm{Log}\,2 - i\pi/2\right]}$$

$$= \boxed{e^{\pi/2 + i\,\mathrm{Log}\,2}}$$

[5]

Question 7: Evaluate $\int_\Gamma \operatorname{Im}((z^2))\,dz$ where Γ is a contour consisting of a line segment from 0 to $1+i$ followed by a line segment from $1+i$ to 1.



γ_1 : $z_1(t) = t(1+i)$, $t: 0 \to 1$

$z_1'(t) = (1+i)$

γ_2 : $z_2(t) = 1+it$, $t: 1 \to 0$

$z_2'(t) = i$

$$\therefore \int_\Gamma \operatorname{Im}(z^2)\,dz = \int_{\gamma_1} + \int_{\gamma_2}$$

$$= \int_0^1 \operatorname{Im}(t+it)^2 (1+i)\,dt + \int_1^0 \operatorname{Im}(1+it)^2\, i\,dt$$

$$= \int_0^1 (2t^2)(1+i)\,dt + \int_1^0 (2t)\,i\,dt$$

$$= 2(1+i)\left[\frac{t^3}{3}\right]_0^1 + \cancel{2}i\left[\frac{t^2}{\cancel{2}}\right]_1^0$$

$$= \frac{2}{3}(1+i) - i$$

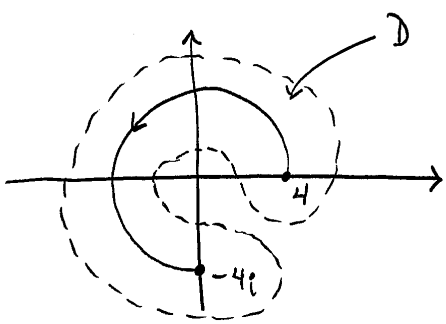
$$= \boxed{\frac{2}{3} - \frac{1}{3}i}$$

[6]

Question 8: Evaluate

$$\int_\Gamma \frac{e^{1/z}}{z^2}\,dz$$

where Γ is the arc of the circle $|z| = 4$ from 4 to $-4i$, proceeding counter-clockwise around the origin.



$f(z) = \dfrac{e^{1/z}}{z^2}$ is continuous on D and has antiderivative $F(z) = -e^{1/z}$ throughout D.

$$\therefore \int_\Gamma \frac{e^{1/z}}{z^2}\,dz = F(-4i) - F(4)$$

$$= -e^{-1/4i} - \left(-e^{1/4}\right)$$

$$= \boxed{e^{1/4} - e^{i/4}}$$

[4]

Question 9: Evaluate

$$I = \int_C \left[\frac{6}{(z-i)^2} + \frac{2}{z-i} + 1 - 3(z-i)^2 \right] dz$$

where C is a positively oriented circle of radius 1 centre $z = i$.

$$I = 6\int_C (z-i)^{-2} dz + 2\int_C (z-i)^{-1} dz + \int_C (z-i)^0 dz - 3\int_C (z-i)^2 dz$$

Recall :
$$\int_C (z-i)^n dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}$$

$$\therefore I = (6)(0) + (2)(2\pi i) + 0 - (3)(0)$$
$$= \boxed{4\pi i}$$

[5]

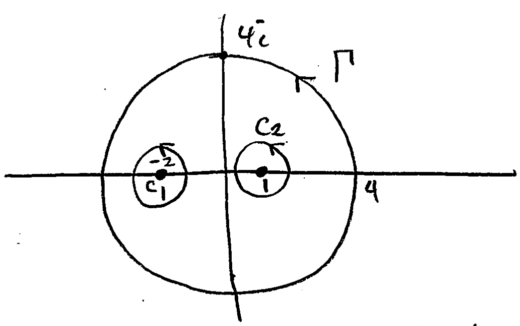
Question 10: Evaluate

$$\int_\Gamma \frac{3z}{(z+2)(z-1)} dz$$

where Γ is the positively oriented circle $|z| = 4$.

$$\frac{3z}{(z+2)(z-1)} = \frac{A}{z+2} + \frac{B}{z-1} \quad \text{where } A = \lim_{z \to -2} \frac{\cancel{(z+2)}\, 3z}{\cancel{(z+2)}(z-1)} = 2$$

$$B = \lim_{z \to 1} \frac{\cancel{(z-1)}\, 3z}{(z+2)\cancel{(z-1)}} = 1$$



$$\therefore I = \left(\int_{C_1} \frac{2}{z+2} dz\right) + \cancel{\left(\int_{C_1} \frac{1}{z-1} dz\right)}^{\to 0} + \cancel{\left(\int_{C_2} \frac{2}{z+2} dz\right)}^{\to 0} + \left(\int_{C_2} \frac{1}{z-1} dz\right)$$

$$= (2)(2\pi i) + (2\pi i)$$
$$= \boxed{6\pi i}$$

[5]