Question 1. [5]: Simplify and express your answer in the form $a + ib$ where a and b are real:

$$\left[\frac{2+i}{6i-(1-2i)}\right]^2$$

$$= \left[\frac{2+i}{-1+8i}\right]^2$$

$$= \left[\frac{(2+i)(-1-8i)}{(-1+8i)(-1-8i)}\right]^2$$

$$= \left[\frac{16-17i}{65}\right]^2$$

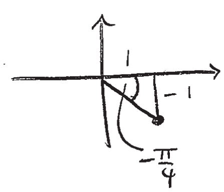
$$= \frac{-253-204i}{65^2}$$

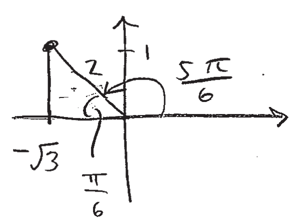
$$= \boxed{\frac{-253}{4225} + \frac{-204}{4225} i}$$

[5]

Question 2. [5]: Determine Arg(z) if $z = (1-i)(-\sqrt{3}+i)$.

$$\underbrace{1-i = \sqrt{2}\, e^{-i\frac{\pi}{4}}}, \quad \underbrace{-\sqrt{3}+i = 2\, e^{i\frac{5\pi}{6}}}$$





$$\therefore \quad (1-i)(-\sqrt{3}+i) = 2\sqrt{2}\, e^{i\left(\frac{5\pi}{6}-\frac{\pi}{4}\right)}$$

$$= 2\sqrt{2}\, e^{i\frac{7\pi}{12}}$$

$$\boxed{\therefore \quad \mathrm{Arg}(z) = \frac{7\pi}{12}}$$

[5]

Question 3. [3]: Express $z = \dfrac{2i}{3e^{4+i}}$ in polar form $re^{i\theta}$.

$$2i = 2e^{i\frac{\pi}{2}}$$

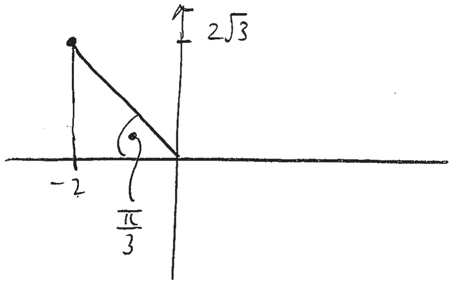
$$\therefore \quad z = \frac{2e^{i\pi/2}}{3e^{4}e^{i}} = \boxed{\underbrace{\left(\frac{2}{3e^{4}}\right)}_{r} e^{\overbrace{i\left(\frac{\pi}{2}-1\right)}^{\theta}}.}$$

[3]

Question 4. [7]: Determine all fourth roots of $-2+2\sqrt{3}i$ and sketch the position of each in the complex plane.

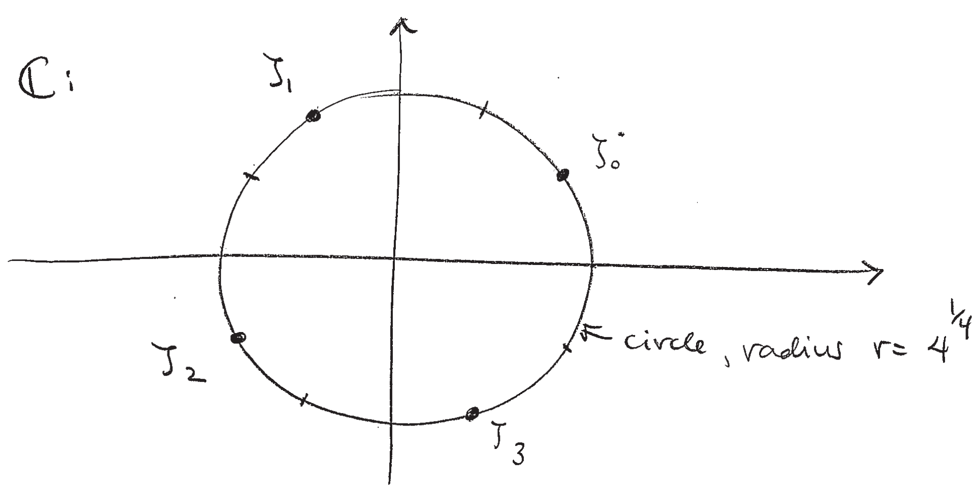
$$-2+2\sqrt{3}\,i = 4e^{i\,2\pi/3} \; ; \quad \therefore \left(-2+2\sqrt{3}\,i\right)^{\frac{1}{4}} = 4^{\frac{1}{4}} e^{\frac{i\left(2\pi/3+2k\pi\right)}{4}}$$

$k = 0,1,2,3.$



$\therefore$ 4^{th} roots are

$$\boxed{\begin{aligned} k=0 &: 4^{\frac{1}{4}} e^{i\pi/6} = z_0 \\ k=1 &: 4^{1/4} e^{i\,2\pi/3} = z_1 \\ k=2 &: 4^{1/4} e^{i\,7\pi/6} = z_2 \\ k=3 &: 4^{1/4} e^{i\,5\pi/3} = z_3 \end{aligned}}$$



[7]

Question 5. [4]: Express $f(z) = \frac{e^{3z}}{|z-1|}$ in the form $u(x,y) + iv(x,y)$ and state the domain of f.

Domain of f is $\{z \in \mathbb{C} \mid z \neq 1\}$.

$$f(z) = \frac{e^{3(x+iy)}}{|x+iy-1|}$$

$$= \frac{e^{3x} e^{i3y}}{\sqrt{(x-1)^2 + y^2}}$$

$$= \boxed{\frac{e^{3x}\cos(3y)}{\sqrt{(x-1)^2+y^2}} + i\,\frac{e^{3x}\sin(3y)}{\sqrt{(x-1)^2+y^2}}}$$

[4]

Question 6. [6]: Let $f(z) = \frac{x^2}{x^2+y^2} + 2i$ where $z = x + iy$ with x and y real. Determine with justification

$$\lim_{z \to 0} f(z)$$

- Let $z \to 0$ along positive real axis, so $y = 0$:

  Then $\frac{x^2}{x^2+y^2} + 2i = \frac{x^2}{x^2} + 2i = 1 + 2i$ as $x \to 0$.

- Let $z \to 0$ along positive imaginary axis, so $x = 0$:

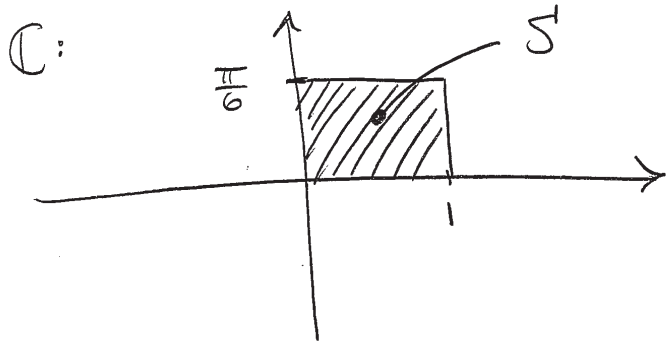
  Then $\frac{x^2}{x^2+y^2} + 2i = \frac{0}{0+y^2} + 2i = 2i$ as $x = 0$

Since these limits differ, $\lim_{z \to 0} f(z)$ $\boxed{\text{does not exist}}$.

[6]

Question 7. [10]: Let $S = \left\{ z \in \mathbb{C} \,\middle|\, 0 \leq \mathrm{Re}(z) \leq 1, 0 \leq \mathrm{Im}(z) \leq \frac{\pi}{6} \right\}$ and $f(z) = e^{2z-1}$.

(i) Sketch S



[2]

(ii) Sketch $f(S)$, the image of S under the mapping f.

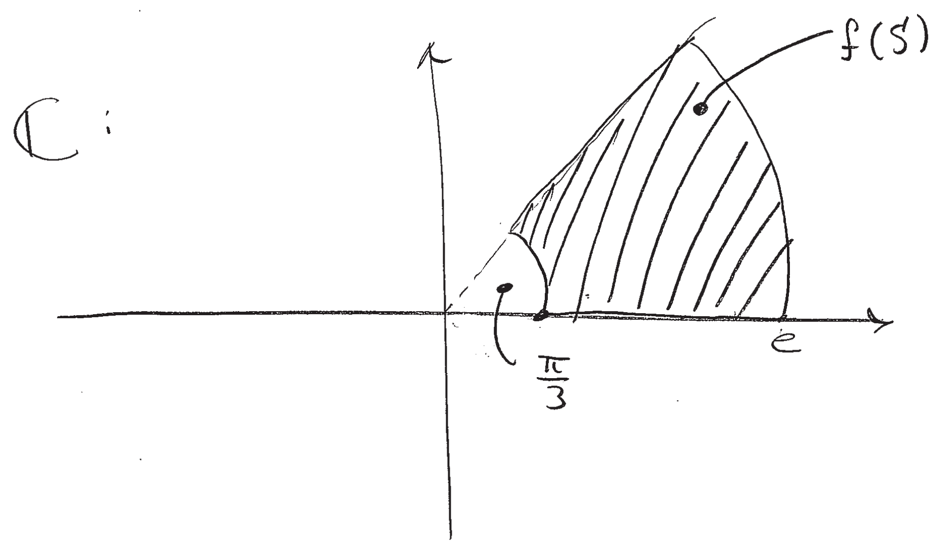
For $x+iy \in S$, $0 \leq x \leq 1$, $0 \leq y \leq \frac{\pi}{6}$.

$f(z) = e^{2z-1} = e^{2(x+iy)-1} = e^{2x-1} e^{i2y}$.

Since $0 \leq x \leq 1$, $-1 \leq 2x-1 \leq 1$, so $e^{-1} \leq e^{2x-1} \leq e$

Since $0 \leq y \leq \frac{\pi}{6}$, $0 \leq 2y \leq \frac{\pi}{3}$.

$\therefore$ $f(S) = \left\{ z \in \mathbb{C} \,\middle|\, z = re^{i\theta} \text{ where } \frac{1}{e} \leq r \leq e, 0 \leq \theta \leq \frac{\pi}{3} \right\}$;



[8]

Question 8. [4]: Determine the points in the complex plane at which $f(z) = x^3+3xy^2-3x+i(y^3+3x^2y-3y)$ is differentiable, (where as usual $z = x + iy$ with x and y real.)

$u(x,y) = x^3+3xy^2-3x$

$v(x,y) = y^3+3x^2y-3y$

$u_x = 3x^2+3y^2-3 \qquad v_y = 3x^2+3y^2-3$

$u_y = 6xy \qquad -v_x = -6xy$

all first partial derivatives continuous.

C.R. equations satisfied only when $6xy = -6xy \Rightarrow 12xy = 0$

$\therefore$ $x=0$ or $y=0$.

$\therefore$ f is differentiable at each point of the real & imaginary axes.

[4]

Question 9. [6]: Let $f(z) = 3x^2 + 2x - 3y^2 - 1 + i(6xy + 2y)$, where $z = x + iy$ with x and y real.

(i) Show that f is entire.

$u = 3x^2+2x-3y^2-1$

$v = 6xy+2y$

$u_x = 6x+2 \qquad v_y = 6x+2$

$u_y = -6y \qquad -v_x = -6y$

Since the C.R. equations are satisfied at each $z \in \mathbb{C}$ and u_x, u_y, v_x, v_y are continuous at each $z \in \mathbb{C}$ then f is differentiable at each $z \in \mathbb{C}$, i.e. f is entire.

[4]

(ii) Express $f(z)$ as a function involving only the variable z.

$x = \frac{z+\bar z}{2}, \quad y = \frac{z-\bar z}{2i}$;

$$f(z) = 3\left(\frac{z+\bar z}{2}\right)^2 + \cancel{2}\left(\frac{z+\bar z}{\cancel{2}}\right) - 3\left(\frac{z-\bar z}{2i}\right)^2 - 1 + i\left[6\left(\frac{z+\bar z}{2}\right)\left(\frac{z-\bar z}{2i}\right) + 2\left(\frac{z-\bar z}{2i}\right)\right]$$

$$= \frac{3}{4}z^2 + \cancel{\frac{3}{2}z\bar z} + \cancel{\frac{3}{4}(\bar z)^2} + z + \cancel{\bar z} + \frac{3}{4}z^2 - \cancel{\frac{3}{2}z\bar z} + \cancel{\frac{3}{4}(\bar z)^2} - 1 + \frac{3}{2}z^2 - \cancel{\frac{3}{2}(\bar z)^2} + z - \cancel{\bar z}$$

$$= \boxed{3z^2 + 2z - 1}$$

[2]