

Math 370 - Complex Analysis

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Elementary Functions of Complex Analysis

Polynomials

- ▶ $p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ where $a_k \in \mathbb{C}$
Here $p(z)$ has **degree** n (assuming $a_n \neq 0$)
- ▶ **Theorem** (*Fundamental Theorem of Algebra*): Every nonconstant polynomial with complex coefficients has at least one zero in $\mathbb{C}$.

Polynomials, cont'd.

- ▶ Consequently, with the help of the division algorithm, any polynomial over $\mathbb{C}$ can be factored into linear factors

$$p(z) = a_n(z - z_1)(z - z_2) \cdots (z - z_n)$$

- ▶ Why? Suppose $p(z)$ has degree n and a zero $z = z_1$. Using the division algorithm,

$$p(z) = (z - z_1)q_1(z) + r(z)$$

where the degree of the quotient factor $q_1(z)$ is $n - 1$, and that of $r(z)$ is strictly less than that of $z - z_1$. That is, $r(z) = k$ for some complex constant k . Now substitute $z = z_1$ to get

$$0 = p(z_1) = (z_1 - z_1)q_1(z_1) + k$$

so that $k = 0$

Polynomials, cont'd.

► So

$$p(z) = (z - z_1)q_1(z)$$

Continue applying this argument to the quotient factor, reducing its degree at each step until it reaches a constant a_n :

$$\begin{aligned} p(z) &= (z - z_1)q_1(z) \\ &= (z - z_1)(z - z_2)q_2(z) \\ &= \dots \\ &= (z - z_1)(z - z_2) \cdots (z - z_n)a_n \end{aligned}$$

Rational Functions

- ▶ Ratios of polynomials:

$$R_{m,n}(z) = \frac{p(z)}{q(z)} = \frac{a_0 + a_1z + a_2z^2 + \cdots + a_mz^m}{b_0 + b_1z + b_2z^2 + \cdots + b_nz^n}$$

- ▶ If $p(z)$ and $q(z)$ have no common factors, the zeros of $q(z)$ are called **poles** of $R_{m,n}(z)$
- ▶ Example: $f(z) = \frac{z^2 + 4}{(z - 2)(z - 3)^2}$ has a pole of order (or multiplicity) 1 at $z = 2$ and a pole of order 2 at $z = 3$.

Partial Fraction Decomposition

- ▶ The partial fraction decomposition result from calculus extends to rational functions over $\mathbb{C}$, but is simplified since all factors of denominator are linear.
- ▶ **Theorem** (*Partial Fraction Decomposition*): Suppose

$$R_{m,n}(z) = \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m}{b_n (z - z_1)^{d_1} (z - z_2)^{d_2} \cdots (z - z_r)^{d_r}}$$

where $d_1 + d_2 + \cdots + d_r = n > m$. Then

$$\begin{aligned} R_{m,n}(z) = & \frac{A_0^{(1)}}{(z - z_1)^{d_1}} + \frac{A_1^{(1)}}{(z - z_1)^{d_1-1}} + \cdots + \frac{A_{d_1-1}^{(1)}}{(z - z_1)} \\ & + \frac{A_0^{(2)}}{(z - z_2)^{d_2}} + \frac{A_1^{(2)}}{(z - z_2)^{d_2-1}} + \cdots + \frac{A_{d_2-1}^{(2)}}{(z - z_2)} \\ & + \cdots + \frac{A_0^{(r)}}{(z - z_r)^{d_r}} + \frac{A_1^{(r)}}{(z - z_r)^{d_r-1}} + \cdots + \frac{A_{d_r-1}^{(r)}}{(z - z_r)} \end{aligned}$$

Example: Partial Fraction Decomposition

Example: Determine the partial fraction decomposition of

$$f(z) = \frac{z^2 + 4}{(z - 2)(z - 3)^2}$$

Solution: Here $\frac{z^2 + 4}{(z - 2)(z - 3)^2} = \frac{a}{z - 2} + \frac{b}{(z - 3)^2} + \frac{c}{z - 3}$

$$\blacktriangleright a = \lim_{z \rightarrow 2} (z - 2)f(z) = \lim_{z \rightarrow 2} \frac{z^2 + 4}{(z - 3)^2} = 8$$

$$\blacktriangleright b = \lim_{z \rightarrow 3} (z - 3)^2 f(z) = \lim_{z \rightarrow 3} \frac{z^2 + 4}{z - 2} = 13$$

$$\begin{aligned}\blacktriangleright c &= \lim_{z \rightarrow 3} \frac{d}{dz} [(z - 3)^2 f(z)] = \lim_{z \rightarrow 3} \frac{d}{dz} \left[\frac{z^2 + 4}{z - 2} \right] \\ &= \lim_{z \rightarrow 3} \left[\frac{(z - 2)(2z) - (z^2 + 4)(1)}{(z - 2)^2} \right] = -7\end{aligned}$$

$$\blacktriangleright \text{Therefore } f(z) = \frac{8}{z - 2} + \frac{13}{(z - 3)^2} + \frac{-7}{z - 3}$$

Coefficients of Partial Fraction Decomposition

In general, if A_k is the coefficient of $\frac{1}{(z - \zeta)^{d-k}}$ in the partial fraction decomposition of the rational function $f(z)$ then

$$A_k = \lim_{z \rightarrow \zeta} \frac{1}{k!} \frac{d^k}{dz^k} \left[(z - \zeta)^d f(z) \right]$$

Exponential, Sine and Cosine Functions

- ▶ Recall: $e^z = e^{x+iy} = e^x[\cos(y) + i \sin(y)]$
- ▶ e^z is entire: $\frac{d}{dz}[e^z] = e^z$
- ▶ Since $e^{z+2\pi i} = e^z e^{2\pi i} = e^z$, say that $e^z (= \exp(z))$ is ***periodic*** with ***period*** $2\pi i$.

Definition of Sine and Cosine Functions

We define

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

Observe: e^z entire $\implies \sin(z)$ and $\cos(z)$ entire.

- ▶ $\frac{d}{dz}[\sin(z)] = \cos(z)$
- ▶ $\frac{d}{dz}[\cos(z)] = -\sin(z)$
- ▶ Many standard real properties still apply:
 $\sin(z + 2\pi) = \sin(z)$, $\sin^2(z) + \cos^2(z) = 1$, etc.
- ▶ But not all! For example, it is not the case that $|\sin(z)| \leq 1$ for all z :

$$|\sin(-2i)| = \left| \frac{e^{i(-2i)} - e^{-i(-2i)}}{2i} \right| > \frac{e^2 - 1}{2} > 1$$

Definitions of other Trigonometric Functions

We define

$$\tan(z) = \frac{\sin(z)}{\cos(z)} \quad \cot(z) = \frac{\cos(z)}{\sin(z)}$$

$$\sec(z) = \frac{1}{\cos(z)} \quad \csc(z) = \frac{1}{\sin(z)}$$

These functions have the same derivatives as their real analogues:

- ▶ $\frac{d}{dz}[\tan(z)] = \sec^2(z)$
- ▶ $\frac{d}{dz}[\cot(z)] = -\csc^2(z)$
- ▶ $\frac{d}{dz}[\sec(z)] = \sec(z)\tan(z)$
- ▶ $\frac{d}{dz}[\csc(z)] = -\csc(z)\cot(z)$

Hyperbolic Functions

We define

$$\sinh(z) = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh(z) = \frac{e^z + e^{-z}}{2}$$

- ▶ Observe: e^z entire $\implies \sinh(z)$ and $\cosh(z)$ entire.
- ▶ $\frac{d}{dz}[\sinh(z)] = \cosh(z)$
- ▶ $\frac{d}{dz}[\cosh(z)] = \sinh(z)$
- ▶ $\sinh(iz) = i \sin(z)$
- ▶ $\cosh(iz) = \cos(z)$