Math 370 - Complex Analysis

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Taylor, Power and Laurent Series

Recap of Last Day

- A series: $\sum_{k=0}^{\infty} c_k = \lim_{n\to\infty} \sum_{k=0}^{n} c_k$

- The geometric series: $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ for $|z| < 1$

Taylor Series

- **Theorem:** If f is analytic in a disk $D = \{|z - z_0| < R\}$, then

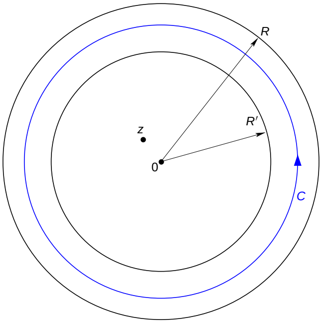
$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j$$

  for every z in D.

  Furthermore, the series converges uniformly in any subdisk $D' = \{|z - z_0| \leq R' < R\}$.

- Consequently, the Taylor series will converge to $f(z)$ everywhere inside the largest disk centred at z_0 over which $f(z)$ is analytic.

Proof in the case $z_0 = 0$



Let C have radius $(R' + R)/2$.
For any z in D',

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta, \quad \text{where}$$

$$= \frac{1}{2\pi i}\int_C f(\zeta)\left[\frac{1}{\zeta}\cdot\frac{1}{1 - z/\zeta}\right]d\zeta \quad \Big\} \text{ notice } |z/\zeta| < 1$$

$$= \frac{1}{2\pi i}\int_C f(\zeta)\left[\frac{1}{\zeta}\cdot\left(\sum_{j=0}^{n}(z/\zeta)^j + \frac{(z/\zeta)^{n+1}}{1 - z/\zeta}\right)\right]d\zeta$$

Proof in the case $z_0 = 0$, continued

Splitting this last expression:

$$\sum_{j=0}^{n} \frac{z^j}{j!} \left(\frac{j!}{2\pi i} \int_C \frac{f(\zeta)}{\zeta^{j+1}}\, d\zeta \right) + \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \cdot \left(\frac{z}{\zeta} \right)^{n+1} d\zeta$$

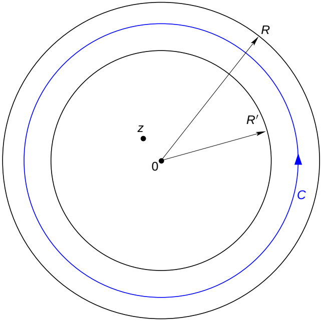
$$= \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} z^j + \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \cdot \left(\frac{z}{\zeta} \right)^{n+1} d\zeta$$

Notice: as $n \to \infty$ the first sum becomes the desired Taylor series.

It remains to show that

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \cdot \left(\frac{z}{\zeta} \right)^{n+1} d\zeta = 0$$

Proof in the case $z_0 = 0$, continued



$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z} \cdot \left(\frac{z}{\zeta}\right)^{n+1} \, d\zeta$$

On C,

$$\left|\frac{1}{\zeta - z}\right| \le \frac{1}{\left(\frac{R+R'}{2} - R'\right)} = \frac{2}{R - R'}$$

and

$$\left|\frac{z}{\zeta}\right|^{n+1} = \frac{|z|^{n+1}}{|\zeta|^{n+1}} \le \left[\frac{R'}{\left(\frac{R'+R}{2}\right)}\right]^{n+1} = \left(\frac{2R'}{R'+R}\right)^{n+1} = \alpha^{n+1}$$

where $\alpha < 1$

Proof in the case $z_0 = 0$, continued

Using these bounds we have

$$\left| \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \cdot \left(\frac{z}{\zeta}\right)^{n+1} \, d\zeta \right|$$

$$\leq \quad \frac{1}{2\pi} \max_{\zeta \in C} |f(\zeta)| \left(\frac{2}{R - R'}\right) \alpha^{n+1}$$

$$\to \quad 0 \ \text{ as } \ n \to \infty$$

Taylor Series Example

Example: Find the Taylor series about $z = 0$ (i.e. the Maclaurin series) for $\text{Log}(1 - z)$ and determine the disk over which is it valid.

Important Taylor (Maclaurin) Series

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \quad \forall z \in \mathbb{C}$$

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \quad \forall z \in \mathbb{C}$$

$$\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots, \quad \forall z \in \mathbb{C}$$

$$\mathrm{Log}(1+z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} z^k}{k} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots, \quad |z| < 1$$

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k = 1 + z + z^2 + z^3 + \cdots, \quad |z| < 1$$

Power Series

- Power series about z_0: $\sum_{j=0}^{\infty} a_j (z - z_0)^j$
- **Theorem:** For each power series there is a real number $0 \leq R \leq \infty$ called the radius of convergence such that the series
  - converges for $|z - z_0| < R$
  - converges uniformly for $|z - z_0| \leq R' < R$
  - diverges for $|z - z_0| > R$

Power Series continued

- As a consequence of the uniform convergence, $f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$ defines an analytic function on the disk $D = \{|z - z_0| < R\}$
- Furthermore, if $f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$, then $a_j = \frac{f^{(j)}(z_0)}{j!}$.

Power Series and Uniform Convergence

- Letting $f_n(z) = \sum_{j=0}^{n} a_j(z - z_0)^j$, a power series is really just $\lim_{n\to\infty} f_n(z)$, and the results on power series follow from:
- **Theorem:** If f_n continuous, $n = 1, 2, \ldots$, and $f_n \to f$ uniformly on T, then f is continuous on T.
- **Theorem:** If f_n continuous, $n = 1, 2, \ldots$, and $f_n \to f$ uniformly on T, then $\int_\Gamma f_n(z)\, dz \to \int_\Gamma f(z)\, dz$ for every Γ in T.
- **Theorem:** If f_n analytic on a simply connected domain D, $n = 1, 2, \ldots$, and $f_n \to f$ uniformly on D, then f is analytic on D.

Operations with Power Series

- **Theorem:** A power series can be integrated and differentiated termwise within its radius of convergence.

- **Theorem:** Suppose $f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$ and $g(z) = \sum_{j=0}^{\infty} b_j (z - z_0)^j$ define analytic functions about z_0, then

  (i) $cf(z) = \sum_{j=0}^{\infty} c a_j (z - z_0)^j$ where c is a constant

  (ii) $(f + g)(z) = \sum_{j=0}^{\infty} (a_j + b_j)(z - z_0)^j$

Operations with Power Series, continued

Theorem: If $f(z) = \sum_{j=0}^{\infty} a_j(z - z_0)^j$ and $g(z) = \sum_{j=0}^{\infty} b_j(z - z_0)^j$ define analytic function about z_0, then fg is analytic at z_0 and

$$(fg)(z) = \sum_{j=0}^{\infty} c_j(z - z_0)^j$$

where

$$c_j = \sum_{k=0}^{j} a_{j-k} b_j$$

Example

Example: Find the Taylor series about $z = 0$ (i.e. the Maclaurin series) for $f(z) = e^{-z^2}$ and state the radius of convergence.

Example

Example: Find the Taylor series about $z = 0$ (i.e. the Maclaurin series) for $f(z) = \dfrac{z}{(1-z)^2}$ and state the radius of convergence.

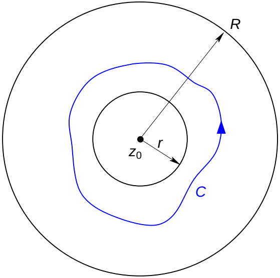
Laurent Series

- **Definition:** A point z_0 is a singularity of f if f is not analytic at z_0 but z_0 is the limit of a sequence of points at which f is analytic.

- For example, $f(z) = \dfrac{e^z}{z-i}$ has a singularity at $z = i$.

- Can we find a Taylor-series-like representation of a function about its singularities?

Laurent Series, continued



Theorem: Suppose f is analytic on the annulus (washer shaped region) $r < |z - z_0| < R$:

Then f can be expressed as

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j} \quad (1)$$

$$= \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j, \quad \text{where...} \quad (2)$$

Laurent Series, continued

- the series (1) converges on $r < |z - z_0| < R$
- convergence is uniform on $r < \rho_1 \leq |z - z_0| \leq \rho_2 < R$, and
- the coefficients a_j are given by

$$a_j = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \, d\zeta$$

  where C is any positively oriented simple closed contour lying inside the annulus and containing z_0.

Laurent Series, continued

Furthermore, if for $r < R$ we have series such that

- $\sum_{j=0}^{\infty} a_j(z - z_0)^j$ converges for $|z - z_0| < R$, and
- $\sum_{j=1}^{\infty} a_{-j}(z - z_0)^{-j}$ converges for $|z - z_0| > r$

then

$$f(z) = \sum_{j=-\infty}^{\infty} a_j(z - z_0)^j$$

defines an analytic function on $r < |z - z_0| < R$ with

$$a_j = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \, d\zeta$$

Laurent Series, continued

- A Laurent series can often be constructed using known series, as opposed to resorting to contour integrals for determining the coefficients.

- For this purpose, it is useful to recall the geometric series for $|z| < 1$:

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j$$