Math 370 - Complex Analysis

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The Residue Theorem

An Important Integral

Recall:

Suppose Γ is a simple closed positively oriented contour, z_0 is inside Γ, and n is an integer. Then for any circular neighbourhood of z_0 (contained in Γ, with positively oriented boundary circle C):

$$\int_\Gamma (z - z_0)^n \, dz = \int_C (z - z_0)^n \, dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if } n \neq -1 \end{cases}$$

Application to Laurent Series

- Suppose Γ is a simple closed positively oriented contour, f is analytic inside and on Γ except at the single isolated singularity z_0 is inside Γ. Then there is some punctured disk $D: \quad 0 < |z - z_0| < R$ inside Γ on which

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j$$

- Suppose D has outer boundary circle C.Then

$$\begin{aligned}
\int_\Gamma f(z)\,dz &= \int_C f(z)\,dz \\
&= \int_C \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j\,dz \\
&= \sum_{j=-\infty}^{\infty} \int_C a_j (z - z_0)^j\,dz \\
&= a_{-1} 2\pi i
\end{aligned}$$

Residues

- **Definition:** If f has an isolated singularity at z_0 then the coefficient a_{-1} of $(z - z_0)^{-1}$ in the Laurent series expansion for f about z_0 is called the residue of f at z_0, and denoted $\mathrm{Res}(f; z_0)$.
- **Example:**

$$f(z) = z^3 \exp(1/z) = z^3 + z^2 + \frac{z}{2!} + \frac{1}{3!} + \frac{1}{4!z} + \frac{1}{5!z^2} + \cdots$$

about the isolated singularity at $z = 0$. So $\mathrm{Res}(f; 0) = 1/4!$

- Using this result with, say, C the positively oriented unit circle:

$$\int_C z^3 e^{1/z}\, dz = 2\pi i[\mathrm{Res}(f; 0)] = \frac{2\pi i}{4!} = \frac{\pi i}{12}$$

Finding Residues

- As previous example shows, one way to find residues of f is to simply work out the Laurent series.
- If z_0 is a removable singularity then the Laurent series contains only non-negative powers of $(z - z_0)$, so $\text{Res}(f; z_0) = a_{-1} = 0$
- If z_0 is a simple pole, then

$$f(z) = \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

so

$$(z - z_0)f(z) = a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \cdots$$

so

$$\lim_{z \to z_0} (z - z_0)f(z) = a_{-1}$$

Finding Residues, continued

If z_0 is pole of order 2, then

$$f(z) = \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

so

$$(z - z_0)^2 f(z) = a_{-2} + a_{-1}(z - z_0) + a_0(z - z_0)^2 + a_1(z - z_0)^3 + \cdots$$

so

$$\frac{d}{dz}\left[(z - z_0)^2 f(z)\right] = a_{-1} + 2a_0(z - z_0) + 3a_1(z - z_0)^2 + \cdots$$

so

$$\lim_{z \to z_0} \frac{d}{dz}\left[(z - z_0)^2 f(z)\right] = a_{-1}$$

Finding Residues, continued

If z_0 is pole of order 3, then

$$f(z) = \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots$$

so

$$(z-z_0)^3 f(z) = a_{-3} + a_{-2}(z-z_0) + a_{-1}(z-z_0)^2 + a_0(z-z_0)^3 + \cdots$$

so

$$\frac{d^2}{dz^2}\left[(z-z_0)^3 f(z)\right] = 2\cdot a_{-1} + 3\cdot 2\cdot a_0(z-z_0) + 4\cdot 3\cdot a_1(z-z_0)^2 + \cdots$$

so

$$\lim_{z\to z_0} \frac{1}{2!}\frac{d^2}{dz^2}\left[(z-z_0)^3 f(z)\right] = a_{-1}$$

Finding Residues, continued

Theorem: If f has a pole of order m at z_0, then

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]$$

The Residue Theorem

- Suppose Γ is a simple closed positively oriented contour and f is analytic inside and on Γ except at the isolated singularities $z_1, z_2, \ldots, z_n$. We wish to evaluate
$$\int_\Gamma f(z)\,dz$$
- Letting $C_1, C_2, \ldots, C_n$ be small circles with centres $z_1, z_2, \ldots, z_n$, respectively, we saw previously that by deforming Γ we have
$$\int_\Gamma f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$$
- But $\displaystyle\int_{C_j} f(z)\,dz = 2\pi i \cdot \text{Res}(f; z_j)$
- So
$$\int_\Gamma f(z)\,dz = 2\pi i \cdot \text{Res}(f; z_1) + 2\pi i \cdot \text{Res}(f; z_2) + \cdots + 2\pi i \cdot \text{Res}(f; z_n)$$

Cauchy's Residue Theorem

Theorem: If Γ is a simple closed positively oriented contour and f is analytic inside and on Γ except at the points $z_1, z_2, \ldots, z_n$ inside Γ, then

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} \text{Res}(f; z_j)$$