Math 370 - Complex Analysis

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Laurent Series

Recap of Last Day: Taylor Series

- **Theorem:** If f is analytic in a disk $D = \{|z - z_0| < R\}$, then

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j$$

  for every z in D.

- The Taylor series will converge to $f(z)$ everywhere inside the largest disk centred at z_0 over which $f(z)$ is analytic.

- So the radius of convergence R is the distance from z_0 to the first point at which f fails to be analytic.

Recap of Last Day: Power Series

- Power series about z_0: $f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$
- converges for $|z - z_0| < R$
- diverges for $|z - z_0| > R$
- R can be found using the ratio test
- $f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$ is analytic on the disk
  $D = \{|z - z_0| < R\}$

Pointwise Convergence Revisited

- Consider a function $F_n(z)$ defined on a set T, where $F_n(z)$ depends on both a non-negative integer n and $z \in \mathbb{C}$.

  For example: $F_n(z) = \sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}$, and T is the disk $|z| < 1$.

- If for any $z \in \mathbb{C}$, $\lim_{n\to\infty} F_n(z)$ exists and equal $F(z)$, we say that F_n converges pointwise to F.

- **Definition:** F_n converges pointwise to F on T if for each $z \in T$, given $\epsilon > 0$ there is a natural number N (possibly depending on both ϵ and z) such that if $n > N$ then $|F_n(z) - F(z)| < \epsilon$.

Pointwise Convergence, Continued

- For $F_n(z) = \sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}$, we saw $F(z) = \frac{1}{1 - z}$, and again T is the disk $|z| < 1$.

- Notice: $|F_n(z) - F(z)| = \left| \frac{z^{n+1}}{1 - z} \right|$ depends on both n and z.

- In order to make this difference small, n must be chosen with reference to the particular z being considered. It is not possible to select a value of n which will make this difference small for every z.

- Here $F_n(z) \to F(z)$ pointwise on T

Uniform Convergence Revisited

- Again consider a function $F_n(z)$ defined on a set T, where $F_n(z)$ depends on both a non-negative integer n and $z \in \mathbb{C}$.

- **Definition:** F_n converges uniformly to F on T if given $\epsilon > 0$ there is a natural number N (possibly depending on ϵ but not on any particular z) such that if $n > N$ then for any $z \in T$, $|F_n(z) - F(z)| < \epsilon$.

- Roughly speaking, if $F_n \to F$ uniformly, for n large enough the difference $|F_n(z) - F(z)|$ will be small regarless of the choice of $z \in T$.

Uniform Convergence, Continued

- Again consider $F_n(z) = \sum_{k=0}^{n} z^k = \dfrac{1 - z^{n+1}}{1 - z}$ and $F(z) = \dfrac{1}{1 - z}$, but this time let T be the disk $|z| < 1/2$.

- Again

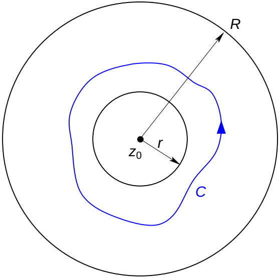
$$|F_n(z) - F(z)| = \left| \frac{z^{n+1}}{1 - z} \right| < \frac{(1/2)^{n+1}}{(1/2)} = \frac{1}{2^n}$$

- Notice: $|F_n(z) - F(z)|$ is bounded by an expression which is independent of z and which goes to zero as $n \to \infty$: $F_n \to F$ uniformly on T.

Laurent Series

- **Definition:** A point z_0 is a singularity of f if f is not analytic at z_0 but z_0 is the limit of a sequence of points at which f is analytic.
- For example, $f(z) = \dfrac{e^z}{z - i}$ has a singularity at $z = i$.
- Can we find a Taylor-series-like representation of a function about its singularities?

Laurent Series, continued



Theorem: Suppose f is analytic on the annulus (washer shaped region) $r < |z - z_0| < R$:

Then f can be expressed as

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j} \quad (1)$$

$$= \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j, \quad \text{where...} \quad (2)$$

Laurent Series, continued

- the series (1) converges on $r < |z - z_0| < R$
- convergence is uniform on $r < \rho_1 \leq |z - z_0| \leq \rho_2 < R$, and
- the coefficients a_j are given by

$$a_j = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \, d\zeta$$

  where C is any positively oriented simple closed contour lying inside the annulus and containing z_0.

Laurent Series, continued

Furthermore, if for $r < R$ we have series such that

- $\sum_{j=0}^{\infty} a_j(z - z_0)^j$ converges for $|z - z_0| < R$, and
- $\sum_{j=1}^{\infty} a_{-j}(z - z_0)^{-j}$ converges for $|z - z_0| > r$

then

$$f(z) = \sum_{j=-\infty}^{\infty} a_j(z - z_0)^j$$

defines an analytic function on $r < |z - z_0| < R$ with

$$a_j = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \, d\zeta$$

Laurent Series, continued

- A Laurent series can often be constructed using known series, as opposed to resorting to contour integrals for determining the coefficients.

- For this purpose, it is useful to recall the geometric series for $|z| < 1$:

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j$$

Example

Example: Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in a Laurent series valid on

(i) $1 < |z| < 3$

(ii) $0 < |z+1| < 2$