Question 1: Compute $\iint_R xy\sqrt{x^2+y^2}\,dA$ where $R = [0,1] \times [0,1]$.

$$I = \frac{1}{2}\int_0^1 x \int_0^1 2y\,\underbrace{(x^2+y^2)}^{1/2}\,dy\,dx$$

$$u = x^2+y^2$$
$$du = 2y\,dy$$

$$= \frac{1}{2}\int_0^1 x\ \frac{2}{3}\Big[(x^2+y^2)^{3/2}\Big]_{y=0}^{1}\,dx$$

$$= \frac{1}{3}\int_0^1 x\Big[(x^2+1)^{3/2} - x^3\Big]\,dx$$

$$= \frac{1}{3}\Big(\frac{1}{2}\Big)\int_0^1 2x\,\underbrace{(x^2+1)}^{3/2}\,dx - \frac{1}{3}\int_0^1 x^4\,dx$$

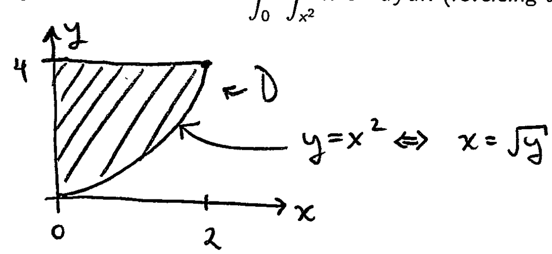
$$u = x^2+1$$
$$du = 2x\,dx$$

$$= \Big(\frac{1}{6}\Big)\Big(\frac{2}{5}\Big)\Big[(x^2+1)^{5/2}\Big]_0^1 - \Big(\frac{1}{3}\Big)\Big(\frac{1}{5}\Big)\Big[x^5\Big]_0^1$$

$$= \frac{1}{15}\big(2^{5/2}-1\big) - \frac{1}{15} = \boxed{\frac{2^{5/2}-2}{15}} = \boxed{\frac{4\sqrt{2}-2}{15}}$$

[5]

Question 2: Calculate $\int_0^2\int_{x^2}^4 x^3 e^{y^3}\,dy\,dx$ (reversing the order of integration may help here.)



$$\therefore I = \int_{y=0}^{4} e^{y^3}\int_{x=0}^{\sqrt{y}} x^3\,dx\,dy$$

$$= \int_0^4 \frac{e^{y^3}}{4}\Big[x^4\Big]_0^{\sqrt{y}}\,dy$$

$$= \frac{1}{4}\Big(\frac{1}{3}\Big)\int_0^4 3y^2\,\underbrace{e^{y^3}}\,dy$$

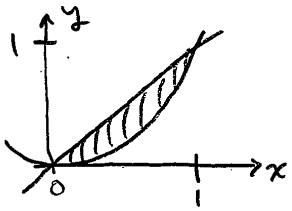
$$u = y^3$$
$$du = 3y^2\,dy$$

$$= \frac{1}{12}\Big[e^{y^3}\Big]_0^4$$

$$= \boxed{\frac{e^{64}-1}{12}}$$

[5]

Question 3: Determine the volume of the solid that lies under the plane $x+2y-z=0$ and above the region in the xy-plane bounded by $y=x$ and $y=x^2$.



$$V = \int_{x=0}^{1}\int_{y=x^2}^{x} x+2y \, dy\, dx$$

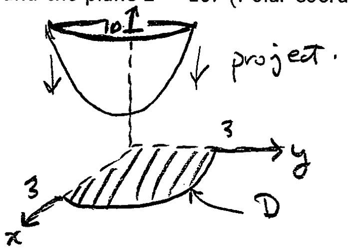
$$= \int_{x=0}^{1} \Big[xy+y^2\Big]_{y=x^2}^{x} dx$$

$$= \int_0^1 x^2+x^2-x^3-x^4 \, dx$$

$$= \left[\frac{2}{3}x^3-\frac{1}{4}x^4-\frac{1}{5}x^5\right]_0^1 = \frac{2}{3}-\frac{1}{4}-\frac{1}{5} = \boxed{\frac{13}{60}}$$

[5]

Question 4: Determine the volume of the solid in the first octant that lies between the paraboloid $z=1+x^2+y^2$ and the plane $z=10$. (Polar coordinates may be helpful here.)



At $z=10$: $10 = 1+x^2+y^2$

$\therefore\ x^2+y^2 = 3^2$

$$\therefore\ V = \iint_D 10-(1+x^2+y^2)\, dA$$

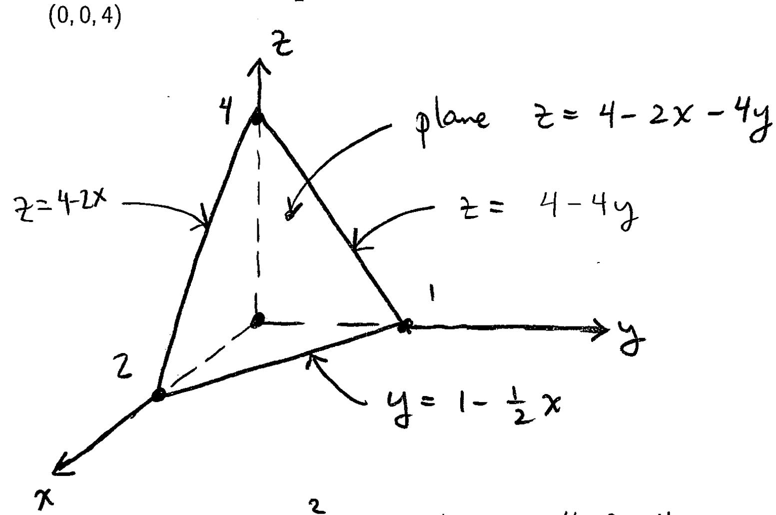
$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{3} (9-r^2)\, r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{9r^2}{2}-\frac{r^4}{4}\right]_0^3 d\theta$$

$$= \int_0^{\pi/2} \left(\frac{81}{2}-\frac{81}{4}\right) - 0 \, d\theta = \frac{\pi}{2}\left(\frac{81}{4}\right) = \boxed{\frac{81\pi}{8}}$$

[5]

Question 5: Evaluate $\iiint\limits_E x\,dV$ where E is the solid tetrahedron with vertices $(0,0,0)$, $(2,0,0)$, $(0,1,0)$ and $(0,0,4)$



$$\therefore\ I = \int_{x=0}^{2} \int_{y=0}^{1-\frac{1}{2}x} \int_{z=0}^{4-2x-4y} x \; dz\, dy\, dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{1-\frac{1}{2}x} x\,(4-2x-4y)\; dy\, dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{1-\frac{1}{2}x} 4x - 2x^2 - 4xy \;\; dy\, dx$$

$$= \int_0^2 \Big[(4x-2x^2)y - 2xy^2\Big]_0^{1-\frac{1}{2}x} dx$$

$$= \int_0^2 (4x-2x^2)(1-\tfrac{1}{2}x) - 2x(1-\tfrac{1}{2}x)^2 \;\; dx$$

$$= \int_0^2 4x - 2x^2 - 2x^2 + x^3 - 2x + 2x^2 - \tfrac{1}{2}x^3 \;\; dx$$

$$= \int_0^2 \tfrac{1}{2}x^3 - 2x^2 + 2x \;\; dx$$

$$= \tfrac{1}{8}\big[x^4\big]_0^2 - \tfrac{2}{3}\big[x^3\big]_0^2 + \big[x^2\big]_0^2$$

$$= 2\,\cancel{\tfrac{1}{8}[16-0]} - \tfrac{2}{3}[8-0] + [4-0] = \boxed{\tfrac{2}{3}}$$

[10]

Question 6: Evaluate $\iiint_E xy^2$ where E is the solid in the first octant that is bounded by the cylinders $x^2+y^2=1$ and $x^2+y^2=4$, and on top by the plane $\underbrace{y+z=3}$.

$z = 3-y = 3 - r\sin\theta$ in polar.

$$I = \int_{\theta=0}^{\pi/2}\int_{r=1}^{2}\int_{z=0}^{3-r\sin\theta} (r\cos\theta)(r\sin\theta)^2\, r\, dz\, dr\, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=1}^{2} r^4\sin^2\theta\cos\theta\,\left[z\right]_0^{3-r\sin\theta} dr\, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=1}^{2} r^4\sin^2\theta\cos\theta\,\left[3-r\sin\theta\right] dr\, d\theta$$

$$= \int_{r=1}^{2}\int_{\theta=0}^{\frac{\pi}{2}} \left[3r^4\sin^2\theta\cos\theta - r^5\sin^3\theta\cos\theta\right] d\theta\, dr$$

$$= \int_{r=1}^{2} \cancel{3}r^4\left[\frac{\sin^3\theta}{\cancel{3}}\right]_0^{\frac{\pi}{2}} - r^5\left[\frac{\sin^4\theta}{4}\right]_0^{\frac{\pi}{2}} dr$$

$$= \int_{r=1}^{2} r^4 - \frac{1}{4}r^5\, dr$$

$$= \frac{1}{5}\left[r^5\right]_1^2 - \frac{1}{24}\left[r^6\right]_1^2$$

$$= \frac{32-1}{5} - \frac{64-1}{24}$$

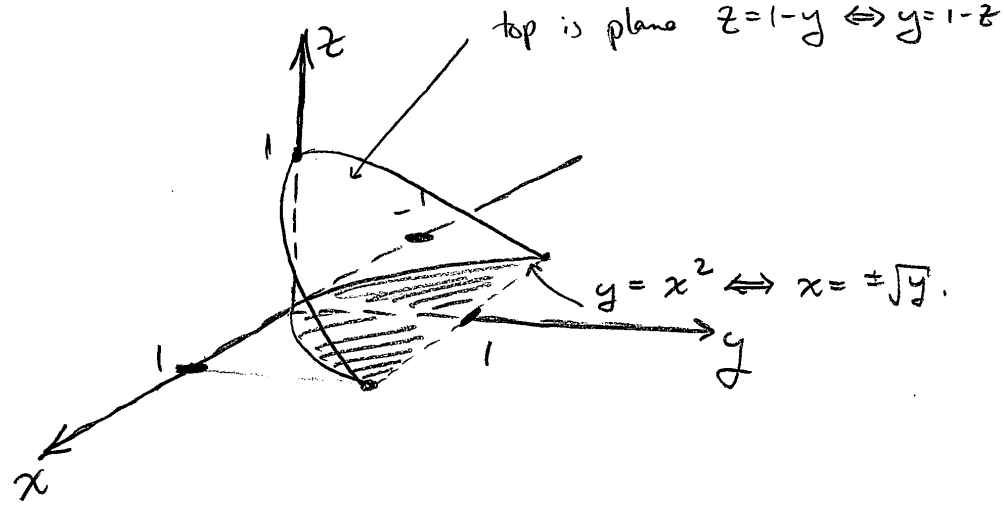
$$= \boxed{\frac{143}{40}}$$

[10]

Question 7: Express the integral

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} f(x,y,z)\,dz\,dy\,dx$$

as an iterated integral in the order $dx\,dy\,dz$



$$\therefore \quad I = \int_{z=0}^{1}\int_{y=0}^{1-z}\int_{x=-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz.$$

[5]