Question 1: A right triangle has perpendicular sides of length $b = 5$ and $h = 12$ m. Let $f(b, h)$ represent the total (i.e. the sum) of the area of the triangle and the length of its hypotenuse. If b is increased by 0.1 and h is decreased by 0.2, use differentials to estimate the resulting change in f.

$$f(b,h) = \frac{1}{2}bh + \sqrt{b^2+h^2}$$

$$df = f_b\,db + f_h\,dh$$

$$= \left[\frac{1}{2}h + \frac{b}{\sqrt{b^2+h^2}}\right]db + \left[\frac{1}{2}b + \frac{h}{\sqrt{b^2+h^2}}\right]dh$$

At $b=5$, $h=12$, $db = 0.1$, $dh = -0.2$,

$$df = \left[\left(\tfrac{1}{2}\right)(12) + \frac{5}{\sqrt{5^2+12^2}}\right](0.1) + \left[\left(\tfrac{1}{2}\right)(5) + \frac{12}{\sqrt{5^2+12^2}}\right](-0.2)$$

$$= \frac{83}{130} - \frac{89}{130} = \boxed{\frac{-3}{65}}$$

[5]

Question 2: Suppose $P = \sqrt{u^2 + v^2 + w^2}$ where

$$u = xe^y, \qquad v = ye^x, \qquad w = e^{xy}$$

Determine $\dfrac{\partial P}{\partial x}$ when $x = 0$ and $y = 2$.

$$\frac{\partial P}{\partial x} = P_u u_x + P_v v_x + P_w w_x$$

$$= \left(\frac{u}{\sqrt{u^2+v^2+w^2}}\right)(e^y) + \left(\frac{v}{\sqrt{u^2+v^2+w^2}}\right)(ye^x) + \left(\frac{w}{\sqrt{u^2+v^2+w^2}}\right)(ye^{xy})$$

At $x=0$, $y=2$: $u = 0$, $v = 2$, $w = 1$, so

$$\frac{\partial P}{\partial x} = (0)(e^2) + \left(\frac{2}{\sqrt{5}}\right)(2) + \left(\frac{1}{\sqrt{5}}\right)(2) = \boxed{\frac{6}{\sqrt{5}}}$$

[5]

Question 3: Find all points on the paraboloid $y = x^2 + z^2$ at which the tangent plane is parallel to the plane $x + 2y + 3z = 1$.

Paraboloid is $\underbrace{y - x^2 - z^2}_{f(x,y,z)} = 0$; normal to $x + 2y + 3z = 1$ is $\langle 1, 2, 3 \rangle$.

At the point in question $\nabla f = k \langle 1, 2, 3 \rangle$ for some constant k

$$\Rightarrow \langle -2x, 1, -2z \rangle = k \langle 1, 2, 3 \rangle$$

$$\Rightarrow x = \left(-\tfrac{1}{2}k\right), \quad 1 = 2k, \quad z = \left(-\tfrac{3}{2}k\right)$$

$$\therefore k = \tfrac{1}{2}, \quad x = -\tfrac{1}{4}, \quad z = -\tfrac{3}{4}$$

and $y = x^2 + z^2 = \left(-\tfrac{1}{4}\right)^2 + \left(-\tfrac{3}{4}\right)^2 = \tfrac{5}{8}$

$\boxed{\therefore \text{ Point is } \left(-\tfrac{1}{4}, \tfrac{5}{8}, -\tfrac{3}{4}\right).}$

[5]

Question 4: The temperature at point $P(x, y, z)$ is given by

$$T = \frac{e^{-3x^2 - 4y^2 + 4z^2}}{4}$$

where x, y, z are in metres. Find (i) the maximum rate of increase of temperature at the point $P_1(2, -1, 2)$ and (ii) the direction in which it occurs (state the direction as a unit vector).

(i) Max is $|\nabla T(2,-1,2)| = \left| \left\langle \frac{-6x\, e^{-3x^2-4y^2+4z^2}}{4}, \frac{-8y\, e^{-3x^2-4y^2+4z^2}}{4}, \frac{8z\, e^{-3x^2-4y^2+4z^2}}{4} \right\rangle \right|$ at $(2,-1,2)$

$$= \left| \left\langle \frac{-12 e^0}{4}, \frac{8 e^0}{4}, \frac{16 e^0}{4} \right\rangle \right|$$

$$= |\langle -3, 2, 4 \rangle|$$

$$= \sqrt{9 + 4 + 16}$$

$$= \boxed{\sqrt{29}}$$

(ii) Max occurs in direction $\nabla T(2,-1,2)$: $\dfrac{\langle -3,2,4 \rangle}{\|\langle -3,2,4 \rangle\|} = \boxed{\dfrac{1}{\sqrt{29}} \langle -3,2,4 \rangle}$

[5]

Question 5: Determine all critical points of $f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$ and classify each as corresponding to a local maximum, a local minimum, or a saddle point.

$f_x = -6x + 6y = 6(y-x) = 0$ ①

$f_y = 6y - 6y^2 + 6x = 6(y+x-y^2) = 0$ ②

① $\Rightarrow$ $y = x$

② $\Rightarrow$ $y + y - y^2 = 0 \Rightarrow y(2-y) = 0 \Rightarrow y = 0,\ y = 2$

$\therefore$ Critical points are $(0,0)$, $(2,2)$

$$D = f_{xx} f_{yy} - (f_{xy})^2 = (-6)(6-12y) - (6)^2$$
$$= 36(2y-1) - 36$$
$$= 72(y-1).$$

C.P.	$D = 72(y-1)$	$f_{xx} = -6$	Conclusion
$(0,0)$	$-72 < 0$	—	saddle point
$(2,2)$	$72 > 0$	$-6 < 0$	local max.

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Question 6: Find the point on the plane $3x + 2y + z = 6$ that is nearest $(0, 0, 0)$. You can use whichever optimization method you like, but be sure to give some explanation of why your solution does indeed correspond to the point nearest $(0, 0, 0)$.

Minimize $L = x^2+y^2+z^2$ subject to $3x+2y+z=6$.

Note: We wish to minimize the distance, which is $\sqrt{L}$, but the (x,y,z) that minimizes L is the same point that minimizes $\sqrt{L}$.

Since $z = 6-3x-2y$,

$$L = x^2+y^2+(6-3x-2y)^2$$

$$L_x = 2x + 2(6-3x-2y)(-3) = 20x+12y-36=0 \quad (1)$$

$$L_y = 2y + 2(6-3x-2y)(-2) = 12x+10y-24=0.$$

$(1) \Rightarrow y = \frac{36-20x}{12} = 3-\frac{5}{3}x$

$(2) \Rightarrow 12x+10\left(3-\frac{5}{3}x\right)-24=0$

$\Rightarrow -\frac{14}{3}x = -6$

$x = \frac{18}{14} = \frac{9}{7}$

$\therefore y = 3-\frac{5}{3}\left(\frac{9}{7}\right) = \frac{6}{7}$

$\therefore z = 6-3\left(\frac{9}{7}\right)-2\left(\frac{6}{7}\right) = \frac{3}{7}$

$\therefore \left(\frac{9}{7}, \frac{6}{7}, \frac{3}{7}\right)$ is nearest $(0,0,0)$

Justification: We know that an abs. min must exist, and it occurs at a critical point. We found a single critical point, so it must correspond to the abs. min

[10]

Question 7: Use the method of Lagrange multipliers to find the absolute maximum value of

$$f(x, y) = 2x + 2y$$

on the ellipse $x^2/16 + y^2/9 = 1$. (This problem maximizes the perimeter of a rectangle that can be inscribed in the ellipse.)

$$g(x,y) = \frac{x^2}{16} + \frac{y^2}{9}$$

$$\left.\begin{array}{l} \nabla f = \lambda \nabla g \\ g(x,y) = 1 \end{array}\right\} \Rightarrow \begin{array}{l} \langle 2, 2 \rangle = \lambda \cdot \langle \frac{x}{8}, \frac{2y}{9} \rangle \\ \frac{x^2}{16} + \frac{y^2}{9} = 1 \end{array}$$

$$\therefore \quad \lambda x = 16 \quad ①$$
$$\lambda y = 9 \quad ②$$
$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \quad ③$$

① & ② $\Rightarrow \lambda \neq 0, \ x \neq 0, \ y \neq 0.$

① ÷ ② $\Rightarrow \dfrac{\cancel{\lambda} x}{\cancel{\lambda} y} = \dfrac{16}{9}$

$\Rightarrow y = \dfrac{9}{16} x.$

③ $\Rightarrow \dfrac{x^2}{16} + \dfrac{1}{9}\left(\dfrac{9}{16} x\right)^2 = 1$

$\dfrac{x^2}{16} + \dfrac{9}{16^2} x^2 = 1$

$x^2 = \dfrac{16^2}{5^2}$

$x = \dfrac{16}{5}, \ \dfrac{-16}{5}$

$\rightarrow x = \dfrac{16}{5} \Rightarrow y = \left(\dfrac{9}{16}\right)\left(\dfrac{16}{5}\right) = \dfrac{9}{5}$

$x = \dfrac{-16}{5} \Rightarrow y = \dfrac{-9}{5}$

$\therefore$ The absolute max.

is $f\left(\frac{16}{5}, \frac{9}{5}\right) = 2\left(\frac{16}{5}\right) + 2\left(\frac{9}{5}\right)$

$= \boxed{10}$

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