Question 1: Determine an equation of the plane containing the points $P_1(3, -1, 1)$, $P_2(4, 0, 2)$ and $P_3(6, 3, 1)$.

Normal $\vec{n} = (\overrightarrow{P_1P_2}) \times (\overrightarrow{P_1P_3})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 3 & 4 & 0 \end{vmatrix}$$

$$= \langle -4, 3, 1 \rangle$$

Using $P_1(3,-1,1)$, equation is

$$(\langle x,y,z\rangle - \langle 3,-1,1\rangle) \cdot \langle -4,3,1\rangle = 0$$

$$-4(x-3) + 3(y+1) + (z-1) = 0$$

or $\quad -4x+3y+z = -14$

or $\quad 4x-3y-z = 14$

[5]

Question 2: A particle starts at the origin with initial velocity $\mathbf{v}_0 = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$. If the particle's acceleration at time t is $\mathbf{a}(t) = 6t\mathbf{i} + 12t^2\mathbf{j} - 6t\mathbf{k}$, find the position of the particle when $t = 2$.

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \int 6t\hat{i} + 12t^2\hat{j} - 6t\hat{k}\ dt$$

$$= 3t^2\hat{i} + 4t^3\hat{j} - 3t^2\hat{k} + \vec{C}_1$$

$$\vec{v}(0) = \hat{i} - \hat{j} + 3\hat{k} \Rightarrow \hat{i} - \hat{j} + 3\hat{k} = 0\cdot\hat{i} + 0\hat{j} + 0\hat{k} + \vec{C}_1$$

$$\Rightarrow \vec{C}_1 = \hat{i} - \hat{j} + 3\hat{k}$$

$$\therefore \vec{v}(t) = (3t^2+1)\hat{i} + (4t^3-1)\hat{j} + (-3t^2+3)\hat{k}$$

$$\therefore \vec{r}(t) = \int \vec{v}(t)\,dt = (t^3+t)\hat{i} + (t^4-t)\hat{j} + (-t^3+3t)\hat{k} + \vec{C}_2$$

$$\vec{r}(0) = \vec{0} \Rightarrow 0\hat{i} + 0\hat{j} + 0\hat{k} + \vec{C}_2 = 0 \Rightarrow \vec{C}_2 = \vec{0}$$

$$\therefore \vec{r}(t) = (t^3+t)\hat{i} + (t^4-t)\hat{j} + (-t^3+3t)\hat{k}$$

$$\therefore \vec{r}(2) = (2^3+2)\hat{i} + (2^4-2)\hat{j} + (-(2)^3 + 3(2))\hat{k}$$

$$= \boxed{10\hat{i} + 14\hat{j} - 2\hat{k}}$$

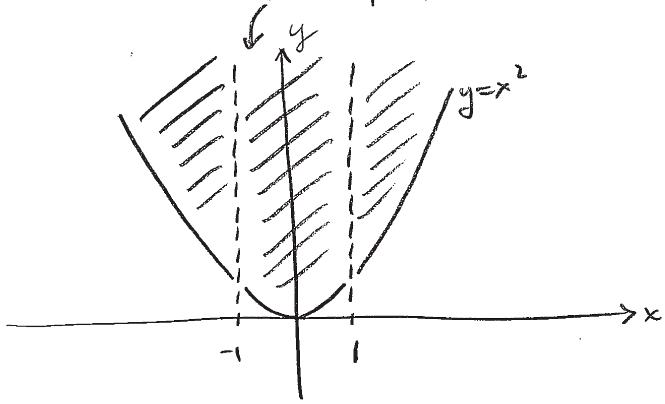
[5]

Question 3: Find and sketch the domain of the function $f(x,y) = \dfrac{\sqrt{y-x^2}}{1-x^2}$.

numerator requires $y - x^2 \geq 0 \Rightarrow y \geq x^2$

denominator requires $1 - x^2 \neq 0 \Rightarrow x \neq \pm 1$

$\therefore$ domain is $\{(x,y) \mid y \geq x^2 \text{ and } x \neq \pm 1\}$.



[5]

Question 4: Show that the following limit does not exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^4+y^4}}$.

- Let $(x,y) \to (0,0)$ along positive x axis, ~~so~~ $y=0$:

  Then $\dfrac{xy}{\sqrt{x^4+y^4}} = \dfrac{0}{\sqrt{x^4+0}} \to 0$ as $x \to 0$.

- Let $(x,y) \to (0,0)$ along line $y=x$,

  then $\dfrac{xy}{\sqrt{x^4+y^4}} = \dfrac{x^2}{\sqrt{2x^4}} = \dfrac{\cancel{x^2}}{\sqrt{2}\,\cancel{x^2}} = \dfrac{1}{\sqrt{2}} \to \dfrac{1}{\sqrt{2}}$ as $x \to 0$.

Since the different paths of approach yield different limiting values,

$\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^4+y^4}}$ does not exist.

[5]

Question 5: Let $f(x,y,z) = x^2 e^{yz}$. Determine $f_{xx}(1,0,1) - f_{zz}(1,0,1)$.

$f_x = 2x e^{yz}$, $f_{xx} = 2e^{yz}$, $f_{xx}(1,0,1) = 2e^{(0)(1)} = 2$

$f_z = x^2 y e^{yz}$, $f_{zz} = x^2 y^2 e^{yz}$, $f_{zz}(1,0,1) = 1^2 \cdot 0^2 \cdot e^{(0)(1)} = 0$

$\therefore \; f_{xx}(1,0,1) - f_{zz}(1,0,1) = 2 - 0 = 2$

[3]

Question 6: Let $f(s,t) = \dfrac{st^2}{s^2+t^2}$. Find at least one point (a,b) at which $f_s(a,b) = 0$.

$$f_s = \frac{(s^2+t^2)(t^2) - (st^2)(2s)}{(s^2+t^2)^2} = \frac{s^2t^2 + t^4 - 2s^2t^2}{(s^2+t^2)^2} = \frac{t^4 - s^2t^2}{(s^2+t^2)^2}$$

$f_s(a,b) = 0 \Rightarrow b^4 - a^2 b^2 = 0$

$\Rightarrow b^2(b^2 - a^2) = 0$

$\Rightarrow b = 0$, a any real number, or $b = \pm a$

$\therefore \; (a,b) = (1,1)$ is one such point.

[3]

Question 7: Use implicit differentiation to find $\dfrac{\partial z}{\partial y}$ if $\sin(xyz) = x + 2y + 3z$.

$$\frac{\partial}{\partial y}\left[\sin((xy)z)\right] = \frac{\partial}{\partial y}\left[x + 2y + 3z\right]$$

$$\cos(xyz)\left[xz + xy\frac{\partial z}{\partial y}\right] = 2 + 3\frac{\partial z}{\partial y}$$

$$\left[xy\cos(xyz) - 3\right]\frac{\partial z}{\partial y} = 2 - xz\cos(xyz)$$

$$\frac{\partial z}{\partial y} = \frac{2 - xz\cos(xyz)}{xy\cos(xyz) - 3}$$

[4]

Question 8: Find the equation of the tangent plane to $z = e^{x^2-y^2}$ at the point where $x = 1$ and $y = -1$.

At $x=1,\ y=-1$ $\quad z = e^{1^2-(-1)^2} = e^0 = 1$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,-1)} = \left.2xe^{x^2-y^2}\right|_{(1,-1)} = 2$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,-1)} = -2ye^{x^2-y^2} = +2$$

$$\therefore\quad z = z_0 + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$z = 1 + 2(x-1) + 2(y+1)$$

$$\therefore\quad 2x + 2y - z = -1$$

[5]

Question 9: For the function $f(x, y)$, a linear approximations at $(1, 1)$ was used to find $f(1.1, 1.1) - f(1, 1) \approx 3$ and $f(1.1, 0.9) - f(1, 1) \approx 2$. Determine $f_x(1, 1)$.

$f(1.1, 1.1) - f(1,1) \approx f_x(1,1)\Delta x + f_y(1,1)\Delta y$ where $\Delta x = 0.1$, $\Delta y = 0.1$

$$\therefore\quad 3 = f_x(1,1)(0.1) + f_y(1,1)(0.1) \quad ①$$

$f(1.1, 0.9) - f(1,1) \approx f_x(1,1)\Delta x + f_y(1,1)\Delta y$ where $\Delta x = 0.1, \Delta y = -0.1$

$$\therefore\quad 2 = f_x(1,1)(0.1) + f_y(1,1)(-0.1) \quad ②.$$

Adding ① & ② :

$$5 = 2 f_x(1,1)(0.1)$$

$$\text{so}\quad f_x(1,1) = \frac{5}{(2)(0.1)} = \boxed{25}$$

[5]

Question 10: Two particles move through space according to the positions functions $\mathbf{r}_1(\mathbf{t}) = \langle t^2, 7t-12, t^2\rangle$ and $\mathbf{r}_2(\mathbf{t}) = \langle 4t-3, t^2, 5t-6\rangle$, where $t \geq 0$ is time. Determine when, if ever, the particles collide.

Solve $\vec{r}_1(t) = \vec{r}_2(t) \Rightarrow$

$$t^2 = 4t-3 \quad (1)$$
$$7t-12 = t^2 \quad (2)$$
$$t^2 = 5t-6 \quad (3)$$

Collision occurs if (1),(2),(3) have a simultaneous solution.

Solving (1):
$$t^2-4t+3=0$$
$$(t-3)(t-1)=0$$
$$t=3,\ t=1$$

$t=3$ satisfies both (2) & (3) (& (1)), but $t=1$ does not.

$\therefore$ Particles collide when $t=3$.

[5]

Question 11: Find the point, if any, at which the space curve $\mathbf{r}(\mathbf{t}) = \langle e^t, e^{2t}, e^{3t}\rangle$ intersects the surface $z = x^2 + 2y$.

$$e^{3t} = (e^t)^2 + 2(e^{2t})$$
$$e^{3t} = e^{2t} + 2e^{2t}$$
$$e^{3t} = 3e^{2t}$$
$$\frac{e^{3t}}{e^{2t}} = 3$$
$$e^t = 3$$
$$\therefore t = \ln 3.$$

$$\vec{r}(\ln 3) = \langle e^{\ln 3}, (e^{\ln 3})^2, (e^{\ln 3})^3 \rangle$$
$$= \boxed{\langle 3, 9, 27\rangle.}$$

[5]