Question 1. [10]:

(a) Determine the shortest distance from the point $(3, 7, -5)$ to the x-axis.

Point on x-axis has form $(x, 0, 0)$.

Distance to point is $\sqrt{(x-3)^2 + (7-0)^2 + (-5-0)^2}$.

Distance is shortest when $x = 3$, giving

$$\text{distance} = \sqrt{7^2 + (-5)^2} = \boxed{\sqrt{74}}$$

[2]

(b) Determine if the following points all lie on the same line: $P_1(0, -5, 5)$, $P_2(1, -2, 4)$, $P_3(3, 4, 2)$.

$$\left|\overrightarrow{P_1P_2} \times \overrightarrow{P_2P_3}\right| = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & -1 \\ 2 & 6 & -2 \end{vmatrix} = \left|0\hat{\imath} - 0\hat{\jmath} + 0\vec{k}\right| = 0$$

$\therefore$ points $\underline{\underline{\text{do}}}$ lie on same line.

[3]

(c) Find an equation of the largest sphere with centre $(5, 4, 9)$ that is completely contained in the first octant. (The first octant is the region of $\mathbb{R}^3$ with $x \geq 0$, $y \geq 0$ and $z \geq 0$.)

Largest possible radius is 4, so

equation is $\boxed{(x-5)^2 + (y-4)^2 + (z-9)^2 = 4^2}$

[2]

(d) Write an equation which represents the set of all points (x, y, z) which are equidistant from $(0, 0, 2)$ and the xy-plane.

Distance from (x, y, z) to xy-plane is $|z|$.

$\therefore$ equation is $\sqrt{(x-0)^2 + (y-0)^2 + (z-2)^2} = |z|$

$$\Rightarrow \quad x^2 + y^2 + (z-2)^2 = z^2$$

$$\Rightarrow \quad x^2 + y^2 + \cancel{z^2} - 4z + 4 = \cancel{z^2}$$

$$\Rightarrow \quad \boxed{z = \frac{x^2 + y^2 + 4}{4}}$$

[3]

Question 2. [10]:
For this question use the vectors

$$\mathbf{a} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}, \quad c = \mathbf{j} - 5\mathbf{k}$$

(a) Are vectors **a** and **b** orthogonal?

$$\vec{a} \cdot \vec{b} = \langle 1, 1, -2 \rangle \cdot \langle 3, -2, 1 \rangle$$
$$= 3 - 2 - 2$$
$$= -1$$
$$\neq 0$$

$\boxed{\therefore\ \vec{a}, \vec{b} \text{ not orthogonal.}}$

[2]

(b) Find a vector of length 4 that is orthogonal to both **a** and **b**.

$\vec{u} = 4 \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$ is the required vector.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ 3 & -2 & 1 \end{vmatrix} = \langle -3, -7, -5 \rangle.$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-3)^2 + (-7)^2 + (-5)^2} = \sqrt{83}$$

$$\therefore \vec{u} = \frac{4}{\sqrt{83}} \langle -3, -7, -5 \rangle$$

$$= \boxed{\left\langle \frac{-12}{\sqrt{83}}, \frac{-28}{\sqrt{83}}, \frac{-20}{\sqrt{83}} \right\rangle}$$

[3]

(c) Do **a**, **b** and **c** all lie in the same plane?

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & 1 & -2 \\ 3 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} = (1)(9) - (1)(-15) + (-2)(3) = 18 \neq 0$$

$\boxed{\therefore\ \vec{a}, \vec{b}, \vec{c} \text{ do not all lie in same plane.}}$

[2]

(d) Place the tail of each vector at $(0, 0, 0)$ and consider the triangle formed by the three terminal points. What is the area of this triangle?

Triangle have vertices $P_1(1,1,-2)$, $P_2(3,-2,1)$, $P_3(0,1,-5)$.

$$\therefore \text{Area is } \frac{1}{2} \left| \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} \right|$$

$$= \frac{1}{2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 3 \\ -1 & 0 & -3 \end{vmatrix}$$

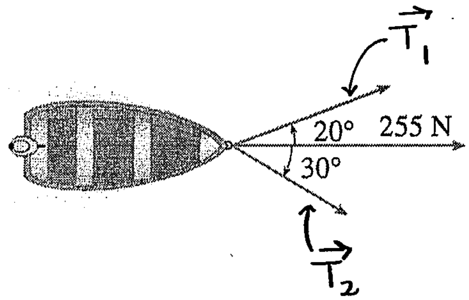
$$= \frac{1}{2} \left| \langle 9, 3, -3 \rangle \right|$$

$$= \frac{1}{2} \sqrt{9^2 + 3^2 + (-3)^2}$$

$$= \boxed{\frac{3\sqrt{11}}{2}}$$

[3]

Question 3. [10]: A boat is pulled into shore using two ropes, as shown in the figure below. If a force of 255 N is needed, find the magnitude of the force in each rope.



① $|\vec{T}_1| \cos(20°)\,\hat{i} + |\vec{T}_2| \cos(30°)\,\hat{i} = 255\,\hat{i}$

② $|\vec{T}_1| \sin(20°)\,\hat{j} - |\vec{T}_2| \sin(30°)\,\hat{j} = 0$

② $\Rightarrow$ $|\vec{T}_1| = |\vec{T}_2| \dfrac{\sin(30)}{\sin(20)}$

Sub ↗ into ①:

$$|\vec{T}_2| \frac{\sin(30)}{\sin(20)} \cos(20) + |\vec{T}_2| \cos(30) = 255$$

$$\therefore\ |\vec{T}_2| = \frac{255}{\left[\frac{\sin(30)}{\sin(20)} \cos(20) + \cos(30)\right]} \doteq \boxed{113.9\ N}$$

$$\therefore\ |\vec{T}_1| = |\vec{T}_2| \frac{\sin(30°)}{\sin(20°)} \doteq \boxed{166.4\ N.}$$

[10]

Question 4. [10]:

(a) Find the (i) vector, (ii) parametric and (iii) symmetric equations of the line through the points $P_1(3,-1,1)$ and $P_2(4,0,2)$

Direction vector is $\vec{v} = \overrightarrow{P_1P_2} = \langle 4-3, 0-(-1), 2-1 \rangle = \langle 1,1,1 \rangle$

using $\vec{r_0} = \langle 3,-1,1 \rangle$, $\vec{r} = \langle x,y,z \rangle$:

$$\boxed{\begin{array}{lll} \text{(i) vector form} & \langle x,y,z\rangle = \langle 3,-1,1\rangle + t\,\langle 1,1,1\rangle \\ \text{(ii) parametric form} & x = 3+t,\ y = -1+t,\ z = 1+t \\ \text{(iii) symmetric form} & x-3 = y+1 = z-1 \end{array}}$$

[4]

(b) Find an equation of the line through $(1,0,-1)$ that is parallel to the line $\dfrac{x-4}{3} = \dfrac{y}{2} = z+2$.

$\frac{x-4}{3} = \frac{y}{2} = z+2 = t \Rightarrow x = 4+3t,\ y = 2t,\ z = -2+t$

$\therefore \langle x,y,z\rangle = \langle 4,0,-2\rangle + t\langle 3,2,1\rangle$

$\therefore$ direction vector of line is $\langle 3,2,1\rangle$

$\therefore$ line through $(1,0,-1)$ with this direction vector

is

$$\boxed{\begin{array}{ll} & \langle x,y,z\rangle = \langle 1,0,-1\rangle + t\,\langle 3,2,1\rangle \\ \underline{\text{or}} & x = 1+3t,\ y = 2t,\ z = -1+t \\ \underline{\text{or}} & \frac{x-1}{3} = \frac{y}{2} = z+1 \end{array}}$$

[4]

(c) Determine the point at which the line $x = 2-t$, $y = 1+3t$, $z = 4t$ intersects the xz-plane.

$y=0$ for points on xz-plane

$\therefore y = 1+3t = 0 \Rightarrow t = -\frac{1}{3}$

$\therefore$ point of intersection has $x = 2-(-\frac{1}{3}) = \frac{7}{3}$, $y=0$,

$z = 4(-\frac{1}{3}) = -4/3$

$\boxed{\therefore \text{ point is } \left(\frac{7}{3}, 0, \frac{-4}{3}\right)}$

[2]

Question 5 [10]:

(a) Express the vector $\langle 2,-1,2\rangle$ as the sum of a vector that is orthogonal to $\mathbf{a} = \langle 1,1,1\rangle$ and one that parallel to $\mathbf{a}$.

Let $\vec{u} = \langle 2,-1,2\rangle$.

Then $\vec{u} = \underbrace{\mathrm{proj}_{\vec{a}}\vec{u}}_{\parallel \text{ to } \vec{a}} + \underbrace{(\vec{u} - \mathrm{proj}_{\vec{a}}\vec{u})}_{\perp \text{ to } \vec{a}}$

$$\mathrm{proj}_{\vec{a}}\vec{u} = \left(\vec{u}\cdot\frac{\vec{a}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|}$$

$$= \left(\langle 2,-1,2\rangle\cdot\frac{\langle 1,1,1\rangle}{\sqrt{3}}\right)\frac{\langle 1,1,1\rangle}{\sqrt{3}}$$

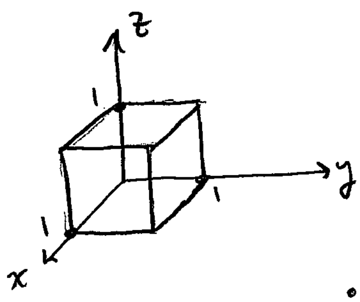
$$= \langle 1,1,1\rangle$$

$\therefore\ \vec{u} - \mathrm{proj}_{\vec{a}}\vec{u} = \langle 2,-1,2\rangle - \langle 1,1,1\rangle = \langle 1,-2,1\rangle$

$$\boxed{\therefore\ \vec{u} = \underbrace{\langle 1,1,1\rangle}_{\parallel \text{ to } \vec{a}} + \underbrace{\langle 1,-2,1\rangle}_{\perp \text{ to } \vec{a}}}$$

[5]

(b) Find the angle between a diagonal of a cube and one of its edges.



Consider diagonal from $(0,0,0)$ to $(1,1,1)$: $\langle 1,1,1\rangle$

consider edge from $(0,0,0)$ to $(1,0,0)$: $\langle 1,0,0\rangle$

Angle between these is θ

$\therefore\ \langle 1,1,1\rangle\cdot\langle 1,0,0\rangle = |\langle 1,1,1\rangle|\,|\langle 1,0,0\rangle| \cos(\theta)$

$$1 = (\sqrt{3})(1)\cos(\theta)$$

$$\therefore\ \cos(\theta) = \frac{1}{\sqrt{3}}$$

$$\theta = \arccos\left(\frac{1}{\sqrt{3}}\right) \doteq \boxed{54.7^\circ}.$$

[5]