Math 370 - Real Analysis

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Real Analysis

What is Real Analysis?

- **Wikipedia:** Real analysis. . . has its beginnings in the rigorous formulation of calculus. It is a branch of mathematical analysis dealing with the set of real numbers. In particular, it deals with the analytic properties of real functions and sequences, including convergence and limits of sequences of real numbers, the calculus of the real numbers, and continuity, smoothness and related properties of real-valued functions.

- **mathematical analysis:** the branch of pure mathematics most explicitly concerned with the notion of a limit, whether the limit of a sequence or the limit of a function. It also includes the theories of differentiation, integration and measure, infinite series, and analytic functions.

In other words…

- In calculus, we learn how to apply tools (theorems) to solve problems (optimization, related rates, linear approximation.)

- In real analysis, we very carefully prove these theorems to show that they are indeed valid.

Thinking back to calculus...

- Most every important concept was defined in terms of limits: continuity, the derivative, the definite integral
- But, the notion of the limit itself was rather vague
- For example,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

  means

  $\sin(x)/x$ gets close to 1 as x gets close to 0.

The key notion

- The key and subtle concept that makes calculus work is that of the limit

- Notion of a limit was truly a major advance in mathematics. Instead of thinking of numbers as only those quantities that could be calculated in a finite number of steps, a number could be viewed as the result of a process, a target reachable after an infinite number of steps.

What makes analysis different

- In the words of the author: "In algebra, we prove equalities directly. That is, we prove that an object (a number perhaps) is equal to another object. In analysis, we generally prove inequalities."

- To illustrate: Suppose x is a real number.

  If $0 \leq x < \epsilon$ for every real number $\epsilon > 0$, then $x = 0$.

  That is, to show that a positive number is zero, it is enough to show that it is less than any other positive real number.

Example of a major result using analysis

Theorem (Fourier): Suppose f is a continuous function defined on the real numbers such that $f(x+2\pi) = f(x)$ for every x, and suppose that f' is also continuous. Then

$$\begin{aligned} f(x) &= \frac{a_0}{2} + [a_1 \cos(x) + b_1 \sin(x)] + [a_2 \cos(2x) + b_2 \sin(2x)] + \\ &\quad [a_3 \cos(3x) + b_3 \sin(3x)] + \cdots \end{aligned}$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt$$

The plan

- **Proofs:** we want to prove things in analysis, so we'll start with a brief review of proof techniques.
- **Set theory:** before we work with sets of real numbers we have to get comfortable with some basic set theory. This will be review for many, but we'll also introduce a few new concepts and see some neat results.
- **Real numbers:** the stars of the show. Limits and all that comes after depend on the structure and properties of real numbers.

The plan, continued

- **Sequences and Series:** properties of lists and sums of real numbers: our first real look at limits.

- **Continuous functions:** important properties of continuous functions which follow from their examination using limits.

- **The Derivative:** We now have the machinery to prove some of the major results: chain rule, mean value theorem, Taylor's theorem.

The plan, continued

- **The Riemann Integral:** Again, defined in terms of limits. Fundamental Theorem of Calculus.

- **Sequences of Functions:** We can extend our study of limits of sequences of real numbers to limits of sequences of functions. The theory is fundamental to many fields: differential equations, harmonic analysis, functional analysis, etc.

Logic and Proofs

Statements: True or False?

- In mathematics we wish to establish whether a given statement is true or false. That is, we wish to establish the statement's truth value.
- **Statement**: a sentence that can be classified as true or false.
- Examples of statements:
  - $2x = 4$ has a solution. True!
  - If f is a continuous function then f is differentiable. False!
  - Every even number greater than 2 can be written as the sum of two prime numbers. Unknown! (Goldbach Conjecture: solve it and you would be famous!)

Compound Statements

- Compound statements are constructed from simple statements using connectives: **not**, **and**, **or**, **if . . . then**, **if and only if**.
- Example: **If** x **and** y are both odd numbers, **then** xy is odd.
- If simple statements of known truth value are combined using connectives, the truth value of the resulting compound statement can be determined.

Not

- **Not:** the logical opposite (negation).
- If p is a true statement, then "*not* p" is false.

  Example: 29 is prime. true

  Example: 29 is not prime. false
- If p is false, then "*not* p" is true.

  Example: 3 is even. false

  Example: It is not the case that 3 is even. true
- "*not* p" sometimes expressed "$\sim p$" .

And

- **And:** conjunction of two statements.

- If p and q are statements, then "*p and q*" is true only when both p and q are true.

  Example: 64 is a square *and* a cube. true

  Example: 9 is a square *and* a cube. false

- "*p and q*" sometimes expressed "$p \wedge q$" .

Or

- **Or:** disjunction of two statements.

- If p and q are statements, then "p *or* q" is true if at least one of p and q is true.

  Example: 9 is a square *or* a cube. true

  Example: 9 is even *or* prime. false

- "p *or* q" sometimes expressed "$p \vee q$" .

If . . . then

- **If** p **then** q**:** implication or conditional statement.

- "*If p then q*" is false when p is true while q is false. It is true in all other cases.

  Example: If n is an integer, then $2n$ is an even number. true

  Example: If n is any integer, then $2n + 1$ is prime. false

  Example: If n is an integer and $n^2 < 0$, then n is prime. true

Other language for conditional statements

"*If p then q*" is equivalent to

- "*p implies q*"
- "*p only if q*"
- "*q if p*"
- "*q provided that p*"
- "*p is sufficient for q*"
- "*q is necessary for p*"
- "$p \implies q$"

If and only if

- *p* **if and only if** *q*: logical equivalence.
- Equivalent to saying that both "*p implies q*" and "*q implies p*"
- true if both *p* and *q* are true, or both *p* and *q* are false.

  Example: $\sqrt{x^2} = x$ if and only if $x \geq 0$. true

  Example: *f* is a differentiable function if and only if *f* is continuous. false
- "*p* if and only if *q*" also written "$p \iff q$"

Negation of compound statements

- **not “***p* **and** *q***”** is equivalent to **“not** *p* **or not** *q***”.**
- **not “***p* **or** *q***”** is equivalent to **“not** *p* **and not** *q***”.**
- **not “***p* **implies** *q***”** is equivalent to **“***p* **and not** *q***”.**

Standard Proof Techniques

Typical situation

- We wish to prove a statement (a theorem maybe) of the form “if p then q”.
- p is called the hypothesis and q the conclusion.
- Example: If n is an even integer greater than 2, then n can be written as a sum of two prime numbers.

Proof as an Art

Steven Lay: The construction of a proof of the implication $p \implies q$ can be thought of as building a bridge of logical statements to connect the hypothesis p with the conclusion q. The building blocks that go into the bridge consist of four kinds of statements:

1. definitions,
2. assumptions,
3. theorems that have been previously established as true, and
4. statements that are logically implied by the earlier statements in the proof.

When actually building the bridge, it may not be at all obvious which blocks to use or in what order to use them. This is where experience is helpful, together with perseverance, intuition, and sometimes a good bit of luck.

Direct Proof

- To prove "if p then q", assume statement p is true and use facts, theorems and logic to show that q is true.

- Example:
  Proposition: If n is an even integer then n^2 is also even.

  Proof: Suppose that n is an even integer. Then $n = 2k$ for some integer k. Therefore,

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$$

  Since $2k^2$ is an integer, $2(2k^2)$ is even, so n^2 is even. $\square$

Proof by Contrapositive

- The contrapositive of "if p then q" is "if not q then not p".
- An implication and its contrapositive are logically equivalent. (Why?)
- To prove "if p then q", it is sometimes easier to instead prove "if not q then not p"
- Example:

  Proposition: Suppose n is an integer. If $3n+2$ is even then n is even.
  Proof: We will prove the contrapositive: If n is odd, then $3n+2$ is odd. If n is odd, then $n = 2k+1$ for some integer k. Therefore,

$$\begin{aligned} 3n+2 &= 3(2k+1)+2 \\ &= 6k+5 \\ &= 2(3k+2)+1 \end{aligned}$$

  Since $3k+2$ is an integer, $3n+2$ is odd. □

Proof by Contradiction

- A statement is either true or false. If assuming the statement is false leads to a contradiction of a hypothesis or known fact, then the statement cannot be false, so must be true.

- Example:

  Proposition: There are infinitely many prime numbers.

  Proof: Suppose (for a contradiction) that there are only finitely many prime numbers $p_1 < p_2 < \ldots < p_n$. Consider the positive integer $q = p_1 p_2 \cdots p_n + 1$. q is greater than p_n so cannot be prime, and so is divisible by some prime $p_1, p_2, \ldots, p_n$. But dividing q by any of these primes leaves remainder 1, so q is not divisible by any of $p_1, p_2, \ldots, p_n$. This is a contradiction. Therefore, there are infinitely many prime numbers. □

Counter Examples

- A statement of the form “$p \implies q$" which uses (or implies the use of) the words “for all” or “for every” is a statement about a property that applies to every member of some set.
- To show that such a statement is false, it is enough to produce one example for which p is true but q is false.
- Example:

  Conjecture: For every real number x, if x is irrational, then x^2 is irrational.

  Counerexample: $\sqrt{2}$ is irrational, but $(\sqrt{2})^2 = 2$ is not.

  Note that the conjecture is true for *some* x, for example $x = \sqrt{\pi}$, but not *all* x as claimed. A single counter example is enough to refute the claim.