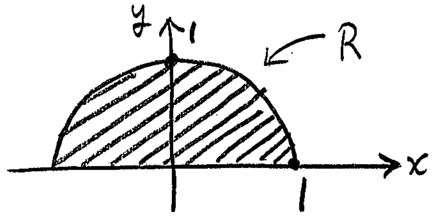
Question 1. [10]:

(a) Let R be the region in the xy-plane that is bounded between the x-axis and the top half of the unit circle (i.e. circle of radius 1 centred at $(0,0)$.) Use polar coordinates to determine the volume of the solid region between R and the surface $z = e^{x^2+y^2}$.



$$V = \iint_R e^{x^2+y^2}\, dA$$

$$= \int_0^{\pi}\int_0^1 e^{r^2}\, r\, dr\, d\theta$$

$0 \le r \le 1$

$0 \le \theta \le \pi$

$x = r\cos\theta$

$y = r\sin\theta$

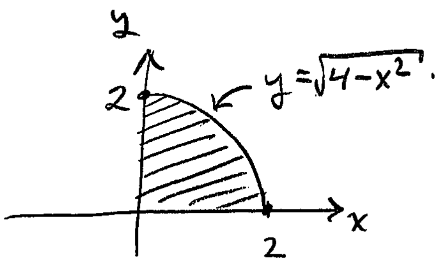
$$= \int_0^{\pi} \left[\frac{e^{r^2}}{2}\right]_0^1 d\theta$$

$$= \frac{e-1}{2}\int_0^{\pi} d\theta$$

$$= \frac{e-1}{2}\left[\theta\right]_0^{\pi} = \boxed{\pi\left(\frac{e-1}{2}\right)}$$

[5]

(b) Evaluate $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} (x^2+y^2)^{3/2}\, dy\, dx$



$$\therefore I = \int_0^{\frac{\pi}{2}}\int_0^2 (r^2)^{3/2}\, r\, dr\, d\theta$$

$0 \le r \le 2$

$0 \le \theta \le \frac{\pi}{2}$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^5}{5}\right]_0^2 d\theta$$

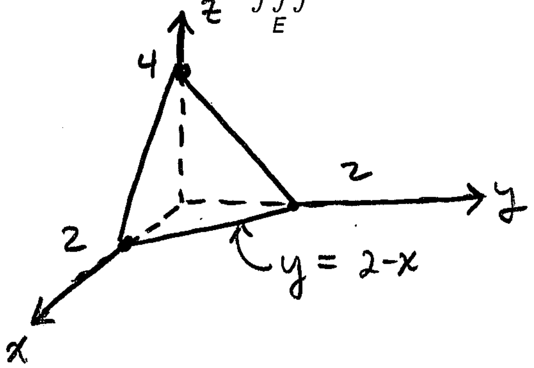
$$= \frac{32}{5}\left[\theta\right]_0^{\pi/2}$$

$$= \boxed{\frac{32\pi}{10}}$$

[5]

Question 2. [10]:

(a) Compute $\iiint_E y\,dV$ where E is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x+2y+z = 4$.



$$I = \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} y \; dz\,dy\,dx$$

$$= \int_0^2 \int_0^{2-x} y\,(4-2x-2y)\,dy\,dx$$

$$= \int_0^2 \left((4-2x)\left[\frac{y^2}{2}\right]_0^{2-x} - \frac{2}{3}\left[y^3\right]_0^{2-x} \right) dx$$

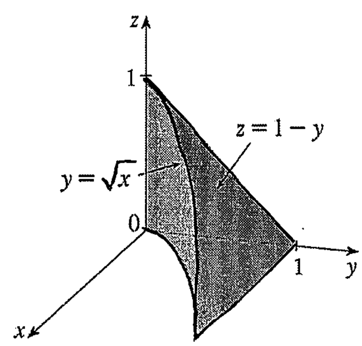
$$= \int_0^2 \frac{1}{2}(4-2x)(2-x)^2 - \frac{2}{3}(2-x)^3 \; dx$$

$$= \int_0^2 \frac{(2-x)^3}{3}\,dx$$

$$= -\frac{1}{12}\left[(2-x)^4\right]_0^2 = \boxed{\frac{4}{3}}$$

[7]

(b) Consider the triple integral $\mathcal{I} = \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\,dz\,dy\,dx$, where the region of integration is



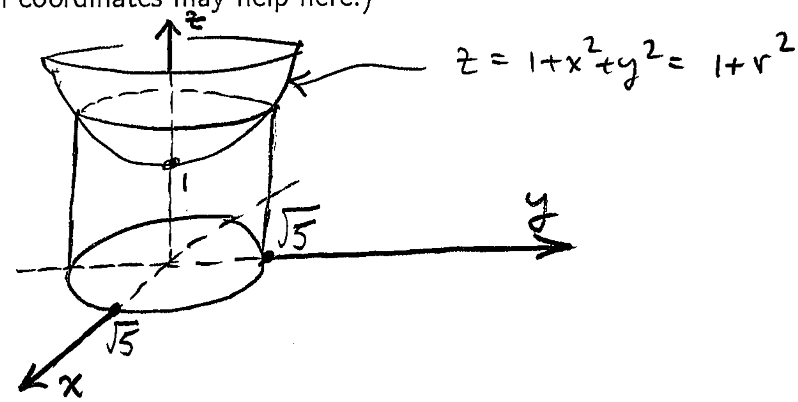
Fill in the appropriate bounds of integration to give the equivalent iterated integral:

$$\mathcal{I} = \int_{\boxed{y=0}}^{\boxed{1}} \int_{\boxed{z=0}}^{\boxed{1-y}} \int_{\boxed{x=0}}^{\boxed{y^2}} f(x, y, z)\,dx\,dz\,dy$$

[3]

Question 3. [10]: Evaluate $\iiint_E e^z\,dV$ where E is enclosed by the paraboloid $z = 1 + x^2 + y^2$, the cylinder $x^2 + y^2 = 5$, and the xy-plane. (Cylindrical coordinates may help here.)

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{array}\right\} \begin{array}{l} 0 \le r \le \sqrt{5} \\ 0 \le \theta \le 2\pi \end{array}$$



$$\iiint_E e^z\,dV = \int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{5}}\int_{z=0}^{1+r^2} e^z\, r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{5}} \left[e^z\right]_0^{1+r^2} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{5}} \left[e^{1+r^2}\cdot r - r\right] dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{e^{1+r^2}}{2} - \frac{r^2}{2}\right]_0^{\sqrt{5}} d\theta$$

$$= \left(\frac{e^6 - 5 - e + 0}{2}\right)\left[\theta\right]_0^{2\pi}$$

$$= \boxed{\pi\left(e^6 - e - 5\right).}$$

[10]

Question 4. [10]: Evaluate $\iiint_H (x^2+y^2)\,dV$ where H is the solid region in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) which is enclosed between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=9$. (Spherical coordinates may help here.)

$H:\ 2 \leq \rho \leq 3, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \frac{\pi}{2}$

$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho\cos\phi.$

$\therefore\ x^2+y^2 = \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta = \rho^2 \sin^2\phi.$

$$\therefore \iiint_H (x^2+y^2)\,dV = \int_{\theta=0}^{\frac{\pi}{2}} \int_{\phi=0}^{\frac{\pi}{2}} \int_{\rho=2}^{3} \rho^2 \sin^2\phi\ \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left[\frac{\rho^5}{5}\right]_2^3 \sin^3\phi\ d\phi\, d\theta$$

$$= \left(\frac{3^5-2^5}{5}\right)\left(\int_0^{\frac{\pi}{2}} d\theta\right)\left(\int_0^{\frac{\pi}{2}} (1-\cos^2\phi)\sin\phi\, d\phi\right)$$

$$= \left(\frac{3^5-2^5}{5}\right)\left(\frac{\pi}{2}\right)\left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{3^5-2^5}{5}\right)\left(\frac{\pi}{2}\right)\left(1-\frac{1}{3}\right)$$

$$= \frac{2\pi(3^5-2^5)}{30} = \boxed{\frac{211\,\pi}{15}}$$

[10]

Question 5 [10]: For this question we will use a transformation to evaluate $\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$ where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$, $(0,1)$.

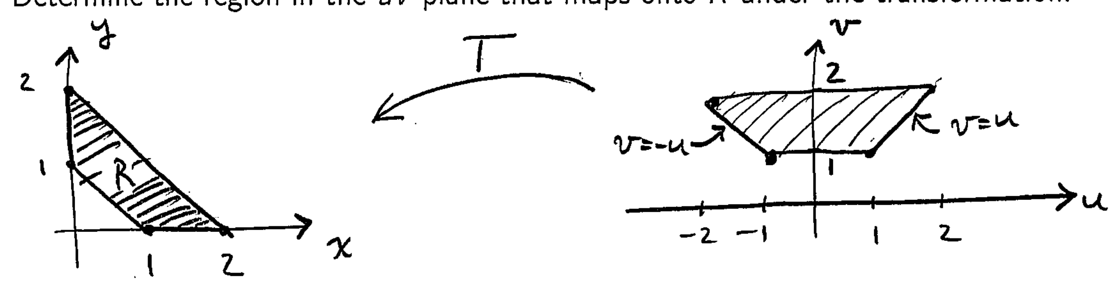
(a) The transformation is $u = y - x$, $v = y + x$. Determine $\frac{\partial(x,y)}{\partial(u,v)}$, the Jacobian of the transformation.

$$\left.\begin{matrix} u = y - x \\ v = y + x \end{matrix}\right\} \Rightarrow \left.\begin{matrix} \frac{u+v}{2} = y \\ \frac{v-u}{2} = x \end{matrix}\right\} \Rightarrow \det\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$$

$$= \det\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \boxed{-\frac{1}{2}}$$

[2]

(b) Determine the region in the uv-plane that maps onto R under the transformation.



(x,y)	(u,v)
$(1,0)$	$(-1,1)$
$(2,0)$	$(-2,2)$
$(0,1)$	$(1,1)$
$(0,2)$	$(2,2)$

[3]

(c)[5] Use parts (a) and (b) to evaluate $\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$.

$$\iint_R \cos\left(\frac{y-x}{y+x}\right) dA = \int_{v=1}^{2} \int_{u=-v}^{v} \cos\left(\frac{u}{v}\right) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

$$= \int_1^2 \int_{-v}^{v} \cos\left(\frac{u}{v}\right)\left(\frac{1}{2}\right) du\, dv$$

$$= \frac{1}{2}\int_1^2 \left[v \sin\left(\frac{u}{v}\right)\right]_{-v}^{v} dv$$

$$= \frac{1}{2}\int_1^2 v\left[\sin(1) - \sin(-1)\right] dv$$

$$= \left(\frac{1}{2}\right) 2\sin(1) \int_1^2 v\, dv$$

$$= \frac{\sin(1)}{2}\left[v^2\right]_1^2$$

$$= \boxed{\frac{3\sin(1)}{2}}$$

[5]