Question 1. [10]:

(a) In what directions is the directional derivative of $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ at the point $P(1,1)$ equal to zero? Use unit vectors to state your answer.

$$\nabla f(1,1) \cdot \langle u, v \rangle = 0$$

$$\Rightarrow \left\langle \frac{(x^2+y^2)(2x) - (x^2-y^2)(2x)}{(x^2+y^2)^2}, \frac{(x^2+y^2)(-2y) - (x^2-y^2)(2y)}{(x^2+y^2)^2} \right\rangle \Bigg|_{(1,1)} \cdot \langle u, v \rangle = 0$$

$$\Rightarrow \langle 1, -1 \rangle \cdot \langle u, v \rangle = 0$$

$$\Rightarrow u - v = 0$$

$\Rightarrow$ $u = v$, and since $\sqrt{u^2+v^2} = 1$,

directions are $\langle u, v \rangle = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ & $\left\langle \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle$.

[5]

(b) Determine the two points (x, y, z) on the hyperboloid $x^2 + 4y^2 - z^2 = 4$ where the tangent plane is parallel to the plane $2x + 2y + z = 5$

Let $f(x,y,z) = x^2 + 4y^2 - z^2 - 4$.

Tangent plane to hyperboloid has normal $\nabla f(x,y,z) = \langle 2x, 8y, -2z \rangle$, and this normal must be parallel to normal to tangent plane, which is $\langle 2, 2, 1 \rangle$.

$\therefore$ $k \langle 2, 2, 1 \rangle = \langle 2x, 8y, -2z \rangle$

$\Rightarrow$ $2k = 2x$, $2k = 8y$, $k = -2z$

$\Rightarrow$ $x = k$, $y = \frac{1}{4}k$, $z = \frac{-1}{2}k$

$\Rightarrow$ $k^2 + \cancel{4\left(\tfrac{1}{4}k\right)^2} - \cancel{\left(\tfrac{-1}{2}k\right)^2} = 4$ since (x,y,z) is on paraboloid

$\Rightarrow$ $k = 2, -2$.

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$\Rightarrow$ Points are $(2, \frac{1}{2}, -1)$, $(-2, \frac{-1}{2}, 1)$.

Question 2. [10]:

(a) Find the absolute maximum and minimum values of $f(x,y) = x^2+y^2-2x$ on the set $D = \{(x,y) \mid x^2+y^2 \le 4\}$, the closed disk of radius 2 and centre $(0,0)$.

$$\left.\begin{aligned} f_x &= 2x-2 \\ f_y &= 2y \end{aligned}\right\} \quad f_x = f_y = 0 \quad \text{at} \quad (x,y) = (1,0), \text{ which is in } D.$$

On the boundary of D, $x^2+y^2=4$, so
$f(x,y) = x^2+y^2-2x = 4-2x$ which is maximized at $(-2,0)$, minimized at $(0,2)$.

Now evaluate:

(x,y)	$f(x,y) = x^2+y^2-2x$
$(1,0)$	-1
$(-2,0)$	8
$(0,2)$	4

$\therefore$ f has an abs. max. of 8 and an abs. min. of -1 on D

[5]

(b) Find the point (x,y,z) on the plane $x+y+z=1$ that is closest to the point $(2,0,-2)$.

Let l = distance from (x,y,z) to $(2,0,-2)$

$$= \sqrt{(x-2)^2+y^2+(z+2)^2}$$

$$= \sqrt{(x-2)^2+y^2+(3-x-y)^2} \quad \text{since } z = 1-x-y$$

It is enough to minimize $l^2 = f(x,y) = (x-2)^2+y^2+(3-x-y)^2$:

$$\left.\begin{aligned} ① \ f_x &= 2(x-2) - 2(3-x-y) = 0 \\ ② \ f_y &= 2y - 2(3-x-y) = 0 \end{aligned}\right\} \Rightarrow 2(x-2) = 2y \Rightarrow x = 2+y$$

Sub. $x = 2+y$ into ①: $\cancel{2}(\cancel{2}+y-\cancel{2}) - \cancel{2}(3-2-y-y) = 0$.

$$\Rightarrow 3y = 1$$
$$\Rightarrow y = 1/3$$
$$\Rightarrow x = 2 + 1/3 = 7/3$$
$$\Rightarrow z = 1 - 7/3 - 1/3 = \frac{-5}{3}$$

$\therefore$ point on plane is $\left(\frac{7}{3}, \frac{1}{3}, \frac{-5}{3}\right)$

[5]

Question 3. [10]: Use the method of Lagrange multipliers to find the absolute maximum and minimum values of $f(x,y) = xy$ on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$. (Note: the ellipse is a closed and bounded set of points, so $f(x,y)$ will certainly have absolute extrema on the ellipse.)

Maximize/minimize $f(x,y) = xy$

subject to $g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} = 1$

$$\nabla f = \lambda \nabla g \Rightarrow \langle y, x \rangle = \lambda \left\langle \frac{2x}{8}, \frac{2y}{2} \right\rangle$$

$$\Rightarrow \left.\begin{array}{l} ① \ y = \frac{\lambda x}{4} \\ ② \ x = \lambda y \\ ③ \ \frac{x^2}{8} + \frac{y^2}{2} = 1 \end{array}\right\}$$

$\lambda \neq 0$, otherwise $x = y = 0$, violating ③

$x \neq 0$, otherwise $y = 0$ also (by ①), again violating ③.

Similarly, $y \neq 0$.

$$① \div ② \Rightarrow \frac{y}{x} = \frac{x}{4y} \Rightarrow x^2 = 4y^2 \Rightarrow x = \pm 2y$$

$$\therefore \text{ by ③ : } \frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1 \Rightarrow \frac{y^2}{2} + \frac{y^2}{2} = 1 \Rightarrow y^2 = 1$$
$$\Rightarrow y = \pm 1$$

$\therefore$ We have points $(2,1), (-2,-1), (-2,1), (2,-1)$.

Evaluating $f(x,y) = xy$ at each of these points we see that f has an abs. max. of 2, and an abs min of -2.

[10]

Question 4. [10]:

(a) Find the volume of the region bounded above by the paraboloid $z = 16 - x^2 - y^2$ and below by the square $R : [0,2] \times [0,2]$ in the xy-plane.

$$V = \int_0^2 \int_0^2 16 - x^2 - y^2 \, dx\, dy$$

$$= \int_0^2 \left[16x - \frac{x^3}{3} - xy^2\right]_0^2 dy$$

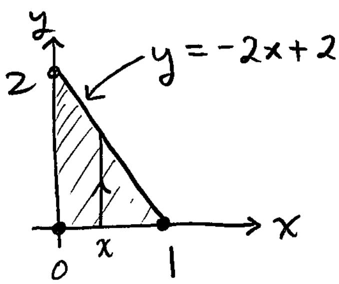
$$= \int_0^2 32 - \frac{8}{3} - 2y^2 \, dy$$

$$= \left[\frac{88}{3}y - \frac{2y^3}{3}\right]_0^2$$

$$= \frac{176}{3} - \frac{16}{3} = \boxed{\frac{160}{3}}$$

[4]

(b) Find the volume of the region bounded above by the paraboloid $z = 16 - x^2 - y^2$ and below by the triangle in the xy-plane with vertices $(0,0)$, $(1,0)$ and $(0,2)$.



$$V = \int_{x=0}^{1} \int_{y=0}^{-2x+2} (16 - x^2 - y^2)\, dy\, dx$$

$$= \int_0^1 \left[16y - x^2y - \frac{y^3}{3}\right]_0^{-2x+2} dx$$

$$= \int_0^1 16(-2x+2) - x^2(-2x+2) - \frac{(-2x+2)^3}{3} \, dx$$

$$= \int_0^1 \left(-32x + 32 + 2x^3 - 2x^2 + \frac{8}{3}x^3 - 8x^2 + 8x - \frac{8}{3}\right) dx$$

$$= \int_0^1 \frac{14}{3}x^3 - 10x^2 - 24x + \frac{88}{3} \, dx$$

$$= \left[\frac{14}{3}\frac{x^4}{4} - \frac{10x^3}{3} - \frac{24x^2}{2} + \frac{88}{3}x\right]_0^1$$

$$= \frac{7}{6} - \frac{20}{6} - \frac{72}{6} + \frac{176}{6} = \boxed{\frac{91}{6}}$$

[6]

Question 5 [10]:

(a) Compute $\iint_R ye^{-xy}\,dA$ where R is the rectangle $R : [0,2] \times [0,3]$.

$$I = \int_0^3 \int_0^2 ye^{-xy}\,dx\,dy$$

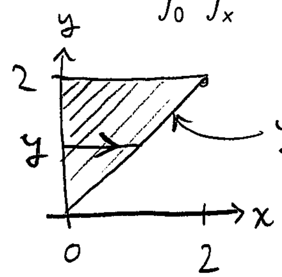
$$= \int_0^3 \left[-e^{-xy}\right]_0^2 dy$$

$$= \int_0^3 \left(-e^{-2y} + 1\right) dy$$

$$= \left[\frac{e^{-2y}}{2} + y\right]_0^3 = \frac{e^{-6}}{2} + 3 - \frac{1}{2} = \boxed{\frac{e^{-6}}{2} + \frac{5}{2}}$$

[5]

(b) Evaluate $\int_0^2 \int_x^2 2y^2 \sin(xy)\,dy\,dx$ (reversing the order of integration may help.)



$$I = \int_{y=0}^{2} \int_{x=0}^{y} 2y^2 \sin(xy)\,dx\,dy$$

$$= \int_0^2 \left[-2y^2 \frac{\cos(xy)}{y}\right]_0^y dy$$

$$= \int_0^2 -2y\cos(y^2) + 2y\,dy$$

$$= \left[-\sin(y^2) + y^2\right]_0^2$$

$$= \boxed{-\sin(4) + 4}$$

[5]