Question 1. [10]:

(a) Determine the unit tangent $\mathbf{T}$ to the space curve $\mathbf{r}(t) = \langle t, 0, 2t - t^2 \rangle$ at the point where $t = 2$.

$$\vec{r}\,'(t) = \langle 1, 0, 2 - 2t \rangle$$

$$\vec{r}\,'(2) = \langle 1, 0, 2 - 2(2) \rangle = \langle 1, 0, -2 \rangle.$$

$$\vec{T} = \frac{\vec{r}\,'(2)}{|\vec{r}\,'(2)|} = \frac{\langle 1, 0, -2 \rangle}{\sqrt{1^2 + (-2)^2}} = \left\langle \frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}} \right\rangle = \boxed{\left\langle \frac{\sqrt{5}}{5}, 0, \frac{-2\sqrt{5}}{5} \right\rangle}$$

[5]

(b) Determine the points (x, y, z) at which the space curve $\mathbf{r}(t) = \langle t, 0, 2t - t^2 \rangle$ intersects the paraboloid $z = x^2 + y^2$.

Solve $\quad 2t - t^2 = t^2 + 0^2$

$$\Rightarrow 2t - 2t^2 = 0$$

$$\Rightarrow 2t(1 - t) = 0$$

$$\Rightarrow t = 0, \; t = 1.$$

$\therefore$ points of intersection are

$$(0, 0, 2(0) - 0^2) = \boxed{(0, 0, 0)}$$

and $(1, 0, 2(1) - 1^2) = \boxed{(1, 0, 1)}$

[5]

Question 2. [10]:

(a) Show that $\lim_{(x,y)\to(0,0)} \frac{5y^4\cos^2 x}{x^4+y^4}$ does not exist.

(i) let $(x,y)\to(0,0)$ along positive x-axis:

$$\lim_{(x,y)\to(0,0)} \frac{5y^4\cos^2 x}{x^4+y^4} = \lim_{x\to 0^+} \frac{5\cdot 0^4\cdot \cos^2 x}{x^4+0^4} = 0$$

(ii) let $(x,y)\to(0,0)$ along positive y-axis:

$$\lim_{(x,y)\to(0,0)} \frac{5y^4\cos^2 x}{x^4+y^4} = \lim_{y\to 0^+} \frac{5\cdot y^4\cdot \cos(0)}{0^4+y^4} = \lim_{y\to 0^+} \frac{5y^4}{y^4} = 5$$

Results of (i) & (ii) differ, so $\lim_{(x,y)\to(0,0)} \frac{5y^4\cos^2 x}{x^4+y^4}$ does not exist.

[5]

(b) Show that $\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$ exists and find the limit.

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}}$$

$$= \lim_{(x,y)\to(0,0)} \underbrace{x^2-y^2}_{\text{continuous!}}$$

$$= \boxed{0}$$

[5]

Question 3. [10]:

(a) Let $z = \frac{x}{y} - \ln(xy)$. Evaluate

$$\left[\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]_{(x,y)=(1,1)}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,1)} = \left[\frac{1}{y} - \frac{y}{xy}\right]_{(1,1)} = 0$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,1)} = \left[\frac{-x}{y^2} - \frac{x}{xy}\right]_{(1,1)} = -1-1 = -2$$

$$\therefore \left[\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]_{(1,1)} = 0-(-2) = \boxed{2}$$

[3]

(b) Let $u(x,t) = e^{-bt}\sin(ax)$ where a and b are constants. Find the realtionship between a and b if $u_t = u_{xx}$.

$$u_t = -be^{-bt}\sin(ax)$$

$$u_x = ae^{-bt}\cos(ax)$$

$$u_{xx} = -a^2e^{-bt}\sin(ax)$$

$$u_t = u_{xx} \Rightarrow -be^{-bt}\sin(ax) = -a^2e^{-bt}\sin(ax)$$

$$\Rightarrow \boxed{b = a^2}$$

[4]

(c) Let $g(x,y,z) = \sqrt{1+xz} + \sqrt{1-xy}$. Find g_{xyz}. (Hint: you may assume that Clairaut's Theorem applies) ~~and so select the most efficient order of differentiation for each term of g.)~~

$$g_{xyz} = \frac{\partial}{\partial x\,\partial z\,\partial y}\left[\sqrt{1+xz}\right] + \frac{\partial}{\partial x\partial y\partial z}\left[\sqrt{1-xy}\right]$$

$$= 0 + 0$$

$$= \boxed{0}$$

[3]

Question 4. [10]:

(a) Find an equation of the tangent plane to $z = \ln(x - 2y)$ at the point where $x = 3$ and $y = 1$.

$f(x,y) = \ln(x-2y)$.

$f(3,1) = \ln(3-2(1)) = 0$.

$f_x(3,1) = \left.\dfrac{1}{x-2y}\right|_{(3,1)} = \dfrac{1}{3-2} = 1$

$f_y(3,1) = \left.\dfrac{-2}{x-2y}\right|_{(3,1)} = \dfrac{-2}{3-2} = -2$.

$\therefore \quad z = f(3,1) + f_x(3,1)(x-3) + f_y(3,1)(y-1)$

$z = 0 + 1\cdot(x-3) + (-2)(y-1)$

$\boxed{x - 2y - z = 1}$

[6]

(b) Let $f(x, y) = \ln(x - 2y)$. Use a linear approximation to estimate $f(2.9, 1.1)$.

From (a): $f(x,y) \approx (x-3) - 2(y-1)$ for (x,y) near $(3,1)$

$\therefore f(2.9, 1.1) \approx (2.9-3) - 2(1.1-1)$

$= -0.1 - 0.2$

$= \boxed{-0.3}$

[4]

Question 5 [10]:

(a) Use the chain rule to determine $\frac{dw}{dt}$ at $t = 0$ if

$$w = e^{x^2+y^2}, \qquad x = \cos t + \sin t, \qquad y = \cos t - \sin t$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= 2xe^{x^2+y^2}\cdot(-\sin t + \cos t) + 2ye^{x^2+y^2}(-\sin t - \cos t)$$

At $t=0$, $x = 1+0 = 1$, $y = 1-0 = 1$, so

$$\left.\frac{dw}{dt}\right|_{t=0} = (2)(1)\,e^{1^2+1^2}\left(-\cancelto{0}{\sin(0)} + \cancelto{1}{\cos(0)}\right) + (2)(1)e^{1^2+1^2}\left(-\cancelto{0}{\sin(0)} - \cancelto{1}{\cos(0)}\right)$$

$$= 2e^2 - 2e^2 = \boxed{0}$$

[5]

(b) Use the chain rule to determine $\frac{\partial w}{\partial v}$ at $(u, v) = (-1, 2)$ if

$$w = xy + \ln z, \qquad x = \frac{v^2}{u}, \qquad y = u + v, \qquad z = \cos u$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

$$= y\cdot\frac{2v}{u} + x\cdot 1 + \cancelto{0}{\frac{1}{z}\cdot 0}$$

At $(u,v) = (-1,2)$ we have $x = \frac{2^2}{-1} = -4$, $y = -1+2 = 1$

$$\therefore \left.\frac{\partial w}{\partial v}\right|_{(-1,2)} = \frac{(1)(2)(2)}{(-1)} + (-4) = \boxed{-8}$$

[5]