**Question 1. [10]:**

**(a)** Determine an equation of the plane through the point $P(3,-1,2)$ that is perpendicular to the $y$-axis.

$$\boxed{y=-1}$$

**[2]**

**(b)** Determine the distance from the point $P(a,b,c)$ to the $xz$-plane.

Horizontal distance to $xz$-plane is simply $|b|$.

**[2]**

**(c)** Suppose **u** is orthogonal to both **v** and **w**. Show that **u** is also orthogonal to $a\mathbf{v}+b\mathbf{w}$ for any choice of scalars $a$ and $b$.

$$\vec{u}\cdot(a\vec{v}+b\vec{w}) = a(\vec{u}\cdot\vec{v})+b(\vec{u}\cdot\vec{w})$$
$$=(a)(0)+(b)(0)$$
$$=0$$
$$\therefore\ \vec{u}\perp a\vec{v}+b\vec{w}.$$

**[3]**

**(d)** Find a vector of length 6 that has the same direction as $\langle -2,4,2\rangle$ .

Vector is $\vec{u}=6\dfrac{\langle -2,4,2\rangle}{|\langle -2,4,2\rangle|}=\left\langle \dfrac{-12}{2\sqrt{6}},\dfrac{24}{2\sqrt{6}},\dfrac{12}{2\sqrt{6}}\right\rangle$

$$=\langle -\sqrt{6},2\sqrt{6},\sqrt{6}\rangle.$$

**[3]**

**Question 2. [10]:**

**(a)** Let $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$. Determine

$$\mathrm{comp}_{\mathbf{v}}\mathbf{u} - \mathrm{comp}_{\mathbf{u}}\mathbf{v}$$

$$= \frac{\langle 2,-1,5\rangle \cdot \langle 1,-5,2\rangle}{\sqrt{1+25+4}} - \frac{\langle 1,-5,2\rangle \cdot \langle 2,-1,5\rangle}{\sqrt{4+1+25}}$$

$$= 0.$$

[3]

**(b)** Determine the angle between the vectors **u** and **v** from part (a).

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos(\theta)$$

$$\Rightarrow \quad \theta = \arccos\left( \frac{\langle 2,-1,5\rangle \cdot \langle 1,-5,2\rangle}{\sqrt{4+1+25}\sqrt{1+25+4}} \right)$$

$$= \arccos\left(\frac{17}{30}\right).$$

[2]

**(c)** Suppose **a** and **b** are orthogonal vectors which define the sides of a rectangle with diagonals $\mathbf{p} = \mathbf{a} + \mathbf{b}$ and $\mathbf{q} = \mathbf{a} - \mathbf{b}$. Show that if the diagonals intersect at right angles then the rectangle is in fact a square.

$$\vec{p} \perp \vec{q} \quad \Rightarrow \quad \vec{p} \cdot \vec{q} = 0$$

$$\Rightarrow \quad (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$$

$$\Rightarrow \quad |\vec{a}|^2 + \cancel{\vec{a} \cdot \vec{b}} - \cancel{\vec{a} \cdot \vec{b}} - |\vec{b}|^2 = 0$$

$$\Rightarrow \quad |\vec{a}|^2 = |\vec{b}|^2$$

$$\Rightarrow \quad |\vec{a}| = |\vec{b}|$$

[5]

**Question 3. [10]:**
For this question use the points $P(2,-2,1)$, $Q(3,-1,2)$ and $R(3,-1,1)$

**(a)** Find a unit vector that is orthogonal to the plane containing the triangle $PQR$ .

$$\vec{u} = \overrightarrow{PQ} = \langle 1,1,1\rangle$$
$$\vec{v} = \overrightarrow{PR} = \langle 1,1,0\rangle$$
$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\hat{\imath}+\hat{\jmath} = \langle -1,1,0\rangle.$$

$\therefore$ unit vector is $\vec{n} = \dfrac{\langle -1,1,0\rangle}{|\langle -1,1,0\rangle|} = \left\langle \dfrac{-1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, 0\right\rangle.$

[4]

**(b)** Determine the area of the triangle $PQR$ .

$$A = \frac{1}{2}\left|\vec{u}\times\vec{v}\right| = \frac{1}{2}\sqrt{(-1)^2+1^2} = \frac{\sqrt{2}}{2}$$

[3]

**(c)** Determine whether triangle $PQR$ is a right angle triangle.

$$\vec{u} = \overrightarrow{PQ} = \langle 1,1,1\rangle$$
$$\vec{v} = \overrightarrow{PR} = \langle 1,1,0\rangle$$
$$\vec{w} = \overrightarrow{QR} = \langle 0,0,-1\rangle$$

$$\vec{u}\cdot\vec{v} = 1+1+0 = 2 \neq 0$$
$$\vec{u}\cdot\vec{w} = 0+0+(-1) = -1 \neq 0$$
$$\vec{v}\cdot\vec{w} = 0+0+0 = 0 \checkmark$$

$\therefore$ $PR \perp QR$, so $PQR$ is a right triangle.

[3]

**Question 4. [10]:**

**(a)** Find an equation of the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$. State both the parametric form and symmetric form of the line.

Direction vector is $\vec{PQ} = \langle 4, -3, 7 \rangle$.

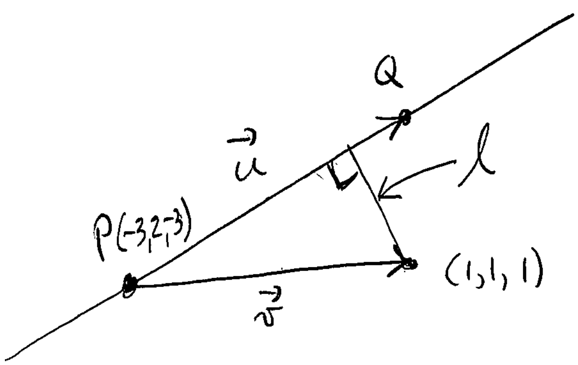
$\vec{r}(t) = \langle -3, 2, -3 \rangle + t \langle 4, -3, 7 \rangle$.

$\therefore \quad x = -3 + 4t, \; y = 2 - 3t, \; z = -3 + 7t \Big\}$ parametric

$\therefore \quad \dfrac{x+3}{4} = \dfrac{-y+2}{3} = \dfrac{z+3}{7} \Big\}$ symmetric.

**[5]**

**(b)** Use projections to find the (shortest) distance from $(1, 1, 1)$ to the line in part (a).




$\vec{v} = \langle 1,1,1 \rangle - \langle -3, 2, -3 \rangle = \langle 4, -1, 4 \rangle$.

$\vec{u} = \vec{PQ} = \langle 4, -3, 7 \rangle$.

$\therefore \operatorname{comp}_{\vec{u}} \vec{v} = \dfrac{\vec{v} \cdot \vec{u}}{|\vec{u}|}$

$= \dfrac{\langle 4, -1, 4 \rangle \cdot \langle 4, -3, 7 \rangle}{|\langle 4, -3, 7 \rangle|}$

$= \dfrac{47}{\sqrt{74}}$

$\therefore l = \sqrt{|\vec{v}|^2 - |\operatorname{comp}_{\vec{u}} \vec{v}|^2}$

$= \sqrt{(16+1+16) - \left(\frac{47}{\sqrt{74}}\right)^2}$

$= \sqrt{\dfrac{233}{74}}$

**[5]**

**Question 5 [10]:**

**(a)** Determine if the line

$$\frac{-x+1}{2} = \frac{y-2}{5} = \frac{-z}{3}$$

is parallel to the plane $2x + y - z = 8$ . Clearly explain your reasoning.

Does line intersect plane?

In parametric form: $t = \frac{-x+1}{2} \Rightarrow x = 1-2t$

$t = \frac{y-2}{5} \Rightarrow y = 2+5t$

$t = -\frac{z}{3} \Rightarrow z = -3t$

Line intersects plane if $2(1-2t) + (2+5t) - (-3t) = 8$ has a sol$^n$.

$\Rightarrow 2 - 4t + 2 + 5t + 3t = 8$

$\Rightarrow 4t = 4$

$\Rightarrow t = 1$

$\therefore x = 1-2(1) = -1$

$y = 2+5(1) = 7$

$z = -3(1) = -3$

So line intersects plane at point $(-1, 7, -3)$, and so line is not parallel to plane. **[5]**

**(b)** Determine the equation of the plane through the point $P(1, 2, 3)$ which is orthogonal to the line of intersections of the planes $M$: $3x - 6y - 2z = 15$ and $N$: $2x + y - 2z = 5$ .

$\vec{n}_M = \langle 3, -6, -2 \rangle$, $\vec{n}_N = \langle 2, 1, -2 \rangle$.

$\therefore$ direction vector of line is $\vec{n}_M \times \vec{n}_N = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix}$

$= \langle 14, 2, 15 \rangle$.

$\therefore$ normal to plane we want is $\vec{n} = \langle 14, 2, 15 \rangle$

$\therefore$ equation of plane is $\langle x-1, y-2, z-3 \rangle \cdot \vec{n} = 0$

$\Rightarrow 14(x-1) + 2(y-2) + 15(z-3) = 0$. **[5]**