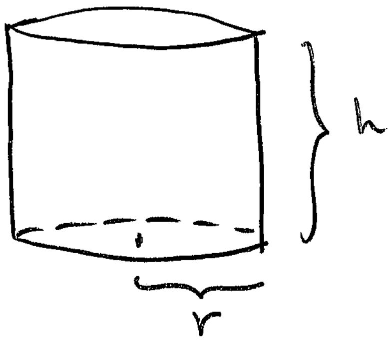
Question 1 [10 points]: The volume of a cylinder is currently 32 cm^3 and is increasing by 4 cm^3/min. The radius is 2 cm and is increasing by 1 cm/min. How fast is the height of the cylinder changing? Give units with your answer. (Recall: the formula for the volume of a cylinder of height h and base radius r is $V = \pi r^2 h$.)



$$\frac{dV}{dt} = +4 \ \frac{cm^3}{min}$$

$$\frac{dr}{dt} = +1 \ \frac{cm}{min}$$

Find $\frac{dh}{dt}$ when $V = 32$ and $r = 2$.

$$V = \pi r^2 h$$

$$\frac{dV}{dt} = \pi \left[2r \frac{dr}{dt} h + r^2 \frac{dh}{dt} \right]$$

When $V = 32$ and $r = 2$ we have $h = \frac{V}{\pi r^2} = \frac{32}{4\pi} = \frac{8}{\pi}$

$$\therefore \quad 4 = \pi \left[(2)(2)(1)\left(\frac{8}{\pi}\right) + (2^2) \frac{dh}{dt} \right]$$

$$4 = 32 + 4\pi \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{-28}{4\pi} = \frac{-7}{\pi} \ \frac{cm}{min}$$

$\therefore$ The height is <u>decreasing</u> by $\frac{7}{\pi} \ \frac{cm}{min}$

[10]

Question 2 [10 points]:

(a) Determine the linearization $L(x)$ of $f(x) = \cos(x) + \ln(1 + 2x)$ at $a = 0$.

$L(x) = f(a) + f'(a)(x-a)$, $a=0$

Here $f(x) = \cos(x) + \ln(1+2x)$; $f(0) = \underbrace{\cos(0)}_{1} + \underbrace{\ln(1+0)}_{0} = 1$

$f'(x) = -\sin(x) + \dfrac{2}{1+2x}$; $f'(0) = -\underbrace{\sin(0)}_{0} + \dfrac{2}{1+0} = 2$

$$\boxed{\therefore\ L(x) = 1 + 2x}$$

[3]

(b) Use a linear approximation to estimate $\cos(-0.1) + \ln(0.8)$ (part (a) should make this easy.)

$\cos(-0.1) + \ln(0.8) \approx L(-0.1) = 1 + 2(-0.1) = \boxed{0.8}$

[2]

(c) A cone has height and base radius equal, so that the volume of the cone is given by $V = \frac{\pi}{3}r^3$. If the radius is measured with a relative error of $1/100$, use differentials (or linear approximation) to ~~determine~~ estimate the relative error in the calculated volume.

$V = \frac{\pi}{3} r^3$

$dV = \frac{\pi}{3} 3r^2\, dr$

$\therefore \dfrac{dV}{V} = \dfrac{\cancel{\frac{\pi}{3}} 3r^2\, dr}{\cancel{\frac{\pi}{3}} r^3} = 3\dfrac{dr}{r}$

$\therefore$ if $\dfrac{dr}{r} = \dfrac{1}{100}$, $\dfrac{dV}{V} = 3\left(\dfrac{1}{100}\right) = \boxed{\dfrac{3}{100}}$

[5]

Question 3 [10 points]:

(a) Determine the intervals of increase and decrease of $f(x) = \dfrac{x^2}{x^4+1}$. State a clear conclusion.

Domain: $(-\infty, \infty)$

$$f'(x) = \frac{(x^4+1)(2x)-(x^2)(4x^3)}{(x^4+1)^2} = \frac{2x^5+2x-4x^5}{(x^4+1)^2} = \frac{2x(1-x^4)}{(x^4+1)^2}$$

$f'(x) = 0$? $x=0$, $x=-1$, $x=1$

$f'(x)$ not exist? no such x.

crit. nums:		-1		0		1	
test. nums:	-2		$-\frac{1}{2}$		$\frac{1}{2}$		2
$f'(x) = \dfrac{2x(1-x^4)}{(x^4+1)^2}$:	$+$	0	$-$	0	$+$	0	$-$
$f(x) = \dfrac{x^2}{x^4+1}$	$\nearrow$	$\frac{1}{2}$	$\searrow$	0	$\nearrow$	$\frac{1}{2}$	$\searrow$

$\therefore$ f is increasing on $(-\infty,-1) \cup (0,1)$,
decreasing on $(-1,0) \cup (1,\infty)$.

[8]

(b) State the relative (or local) extreme values $f(x)$.

f has a rel. max of $\frac{1}{2}$ at $x=-1$ and again at $x=1$;
f has a rel. min of 0 at $x=0$.

[2]

Question 4 [10 points]:

(a) Determine the intervals of concavity of the graph of $f(x) = xe^{-x/2}$. State a clear conclusion.

$$f'(x) = e^{-x/2} - \frac{1}{2} x e^{-x/2} = e^{-x/2}\left(1 - \frac{1}{2}x\right)$$

$$f''(x) = -\frac{1}{2} e^{-x/2}\left(1-\frac{1}{2}x\right) + e^{-x/2}\left(-\frac{1}{2}\right) = -\frac{1}{2}e^{-x/2}\left[2-\frac{1}{2}x\right]$$

- $\underline{f''(x)=0}$? $\quad x = 4$
- $\underline{f''(x) \text{ not exist}}$? no such x.

		4	
test:	0		5
$f''(x) = -\frac{1}{2}e^{-x/2}\left(2-\frac{1}{2}x\right)$:	$-$	0	$+$
$f(x) = xe^{-x/2}$:	$\frown$	0	$\smile$

$\therefore$ graph of $y = f(x)$ is concave down on $(-\infty, 4)$, concave up on $(0, \infty)$.

[8]

(b) State the inflection points (if any) of the graph of $f(x) = xe^{-x/2}$.

Graph has an inflection point at $(0,0)$.

[2]

Question 5 [10 points]: Determine the absolute minimum and maximum values of $f(x) = x\sqrt{4-x^2}$ on the interval $[-1, 2]$. State a clear conclusion.

$f(x) = x(4-x^2)^{\frac{1}{2}}$ } continuous on closed interval $[-1,2]$.

$f'(x) = 1 \cdot (4-x^2)^{\frac{1}{2}} + x\,\frac{1}{2}(4-x^2)^{-\frac{1}{2}} \cdot (-2x)$

$= \sqrt{4-x^2} - \dfrac{x^2}{\sqrt{4-x^2}}$

$= \dfrac{4-x^2-x^2}{\sqrt{4-x^2}}$

$= \dfrac{2(2-x^2)}{\sqrt{4-x^2}}$.

- $f'(x)=0$? $x = \sqrt{2}$, $-\sqrt{2}$ ← outside $[-1,2]$
- $f'(x)$ not exist? $x=2$ ← end point.

x	$f(x) = x\sqrt{4-x^2}$
-1	$-\sqrt{3}$
$\sqrt{2}$	2
2	0

$\therefore$ f has an abs. max. of 2 at $x=\sqrt{2}$.
f has an abs. min. of $-\sqrt{3}$ at $x=-1$.

[10]