Question 1 [8 points]: Differentiate the following functions. It is not necessary to simplify final answers.

(a) $y = 4\sin(\sqrt{x}) = 4\sin(x^{1/2})$

$$y' = 4\cos(x^{1/2}) \cdot \frac{1}{2}x^{-1/2}$$

$$= \frac{2\cos(\sqrt{x})}{\sqrt{x}}$$

[2]

(b) $f(x) = \dfrac{(x^2+3x+2)^4}{2}$

$$f'(x) = \frac{1}{2} \cdot 4(x^2+3x+2)^3\,(2x+3)$$

$$= 2(x^2+3x+2)^3(2x+3).$$

[2]

(c) $y = \ln(x+\cos(x))$

$$y' = \frac{1}{x+\cos(x)} \cdot (1-\sin(x))$$

[2]

(d) $g(t) = \sec(e^t)$

$$g'(t) = \sec(e^t)\tan(e^t) \cdot e^t$$

[2]

Question 2 [12 points]: Differentiate the following functions. It is not necessary to simplify final answers.

(a) $y = \sqrt{1+\cot^2(x)}$ $= \left[1+(\cot(x))^2\right]^{\frac{1}{2}}$

$$y' = \frac{1}{2}\left[1+(\cot(x))^2\right]^{-\frac{1}{2}} 2\cot(x)\ (-\csc^2(x))$$

$$= \frac{-\csc^2(x)\cot(x)}{\sqrt{1+\cot^2(x)}}$$

[3]

(b) $f(x) = \dfrac{1}{1-e^{(x^2)}}$

$$f'(x) = \frac{-1}{(1-e^{x^2})^2}\ (-e^{x^2})\ (2x)$$

$$= \frac{2x\, e^{x^2}}{(1-e^{x^2})^2}$$

[3]

(c) $y = \cos(x\sin(x))$

$$y' = -\sin(x\sin(x))\cdot(\sin(x) + x\cos(x)).$$

[3]

(d) $g(t) = \tan^4(7t^3)$

$$g'(t) = 4\tan^3(7t^3)\cdot\sec^2(7t^3)\cdot 21t^2.$$

$$= 84\, t^2 \tan^2(7t^3)\sec^2(7t^3).$$

[3]

Question 3 [5 points]: Use implicit differentiation to find an equation of the tangent line to the curve $x^4 - x^2y + y^4 = 1$ at the point $(-1, 1)$.

$$\frac{d}{dx}\left[x^4 - x^2y + y^4\right] = \frac{d}{dx}[1]$$

$$4x^3 - 2xy - x^2y' + 4y^3y' = 0$$

At $(x,y) = (-1,1)$:

$$4(-1)^3 - 2(-1)(1) - (-1)^2y' + 4(1)^3y' = 0$$

$$3y' = 2$$

$$y' = \frac{2}{3}$$

$\therefore$ Equation is $y - 1 = \frac{2}{3}(x+1)$ or $y = \frac{2}{3}x + \frac{5}{3}$

[5]

Question 4 [5 points]: Use logarithmic differentiation to find y' where $y = (1+x)^{1/x}$.

$$\ln y = \ln\left[(1+x)^{\frac{1}{x}}\right]$$

$$\ln y = \frac{1}{x}\ln(1+x)$$

$$\frac{1}{y}y' = \left(-\frac{1}{x^2}\right)\ln(1+x) + \left(\frac{1}{x}\right)\left(\frac{1}{1+x}\right)$$

$$\therefore y' = (1+x)^{\frac{1}{x}}\left[\left(\frac{-1}{x^2}\right)\ln(1+x) + \left(\frac{1}{x}\right)\left(\frac{1}{1+x}\right)\right]$$

[5]

Question 5 [5 points]: Find an equation of the tangent line to the curve $y = \log_2(2 + x + x^2)$ at the point where $x = 0$. Simplify all logarithms as much as possible in your final answer.

At $x = 0$ $\quad y = \log_2(2+0+0^2) = 1$

$$y' = \frac{1}{(2+x+x^2)\ln 2} \cdot (1+2x)$$

$$y'\big|_{x=0} = \frac{1}{2\ln(2)}$$

$\therefore$ Equation is $\quad y - 1 = \frac{1}{2\ln(2)}(x-0)$

or $\quad y = \frac{1}{2\ln(2)} x + 1$

[5]

Question 6 [5 points]:

(i) Solve for x: $\ln(x^2 - 1) = 3$

$$x^2 - 1 = e^3$$
$$x^2 = e^3 + 1$$
$$x = \pm\sqrt{e^3+1}.$$

[2]

(ii) Evaluate the limit: $\lim_{x\to 1} e^{-x/(1-x)^2}$

As $x \to 1$, $(1-x)^2 \to 0$, so $\frac{-x}{(1-x)^2} \to -\infty$, so $e^{-x/(1-x)^2} \to 0$

$$\therefore \lim_{x\to 1} e^{-\frac{x}{(1-x)^2}} = 0$$

[3]

Question 7 [5 points]: Let $f(x) = \ln(x + e^{-x})$. Determine $f''(0)$.

$$f'(x) = \frac{1-e^{-x}}{x+e^{-x}}$$

$$f''(x) = \frac{(x+e^{-x})(e^{-x}) - (1-e^{-x})(1-e^{-x})}{(x+e^{-x})^2}$$

$$f''(0) = \frac{\overset{1}{(0+e^0)}\,\overset{1}{(e^0)} - \overset{0}{(1-e^0)}\,\overset{0}{(1-e^0)}}{\underset{1}{(0+e^0)^2}}$$

$$= 1$$

[5]

Question 8 [5 points]: Find all value of x at which tangent lines to $y = x^3 e^x$ are horizontal.

Solve $y' = 0$

$$\Rightarrow \quad 3x^2e^x + x^3e^x = 0$$

$$\Rightarrow \quad x^2e^x(3+x) = 0$$

$$\Rightarrow \quad x = 0, \quad x = -3$$

[5]