Question 1 [10 points]: Evaluate the following limits, if they exist. If a limit does not exist but is ∞ or $-\infty$, state which with an explanation of your answer.

(a) $\displaystyle \lim_{x\to 0} \frac{\tan(7x)}{\sin(4x)}$ $\displaystyle = \lim_{x\to 0} \underbrace{\frac{\sin(7x)}{7x}}_{\to 1} \cdot \underbrace{\frac{1}{\cos(7x)}}_{\to 1} \cdot \underbrace{\frac{4x}{\sin(4x)}}_{\to 1} \cdot \frac{\cancel{7x}}{\cancel{4x}}$

$\displaystyle = \boxed{\frac{7}{4}}$

[4]

(b) $\displaystyle \lim_{x\to\infty} \frac{3x^3+7}{2x^3-x^2+1}$ $\displaystyle = \lim_{x\to\infty} \frac{\cancel{x^3}\left(3+\frac{7}{x^3}\right)}{\cancel{x^3}\left(2-\frac{1}{x}+\frac{1}{x^3}\right)}$

$\displaystyle = \boxed{\frac{3}{2}}$

[3]

(c) $\displaystyle \lim_{x\to 1^+} \frac{\sqrt{3x}(x-1)}{|x-1|}$ $\displaystyle = \lim_{x\to 1^+} \frac{\sqrt{3x}\,\cancel{(x-1)}}{\cancel{(x-1)}}$ since $x-1 \geq 0$ as $x \to 1^+$

$= \boxed{\sqrt{3}}$

[3]

Question 2 [5 points]: Use the intermediate value theorem to show that the equation $x = \cos(x)$ has a solution on the interval $(0, \pi/2)$.

let $f(x) = x - \cos(x)$, continuous on $(-\infty, \infty)$

$f(0) = -1$, $f(\frac{\pi}{2}) = \frac{\pi}{2}$, and $f(0) < 0 < f(\frac{\pi}{2})$

$\therefore$ by the IVT, $f(c) = 0$ for some $0 < c < \frac{\pi}{2}$,

i.e. $c = \cos(c)$.

[5]

Question 3 [5 points]: Let $f(x) = \dfrac{1}{5x-2}$. Use the definition of the derivative to find $f'(x)$. (No credit will be given if $f'(x)$ is found by any other method.)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{1}{5(x+h)-2} - \frac{1}{5x-2}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{(\cancel{5x} \cancel{-2}) - (\cancel{5x} + 5h \cancel{-2})}{[5(x+h)-2][5x-2]}\right]$$

$$= \lim_{h \to 0} \left(\frac{1}{\cancel{h}}\right) \cdot \frac{(-5\cancel{h})}{(5(x+h)-2)(5x-2)}$$

$$= \boxed{\frac{-5}{(5x-2)^2}}$$

[5]

Question 4 [10 points]: Differentiate the following functions. It is not necessary to simplify your answers.

(a) $y = 7x^4 - 4x^3 + 2\pi^2$

$$\boxed{y' = 28x^3 - 12x^2}$$

[3]

(b) $y = \cos(5\pi) - \frac{\sqrt{5}}{\sqrt{x}}$ $= \cos(5\pi) - \sqrt{5}\, x^{-1/2}$

$$\boxed{y' = \frac{\sqrt{5}}{2}\, x^{-3/2}}$$

[4]

(c) $f(x) = \frac{x^{-2}}{2} - \frac{1}{x}$ $= \frac{1}{2} x^{-2} - x^{-1}$

$$\boxed{f'(x) = -x^{-3} + x^{-2}}$$

[3]

Question 5 [6 points]: Differentiate the following functions. It is not necessary to simplify your answers.

(a) $f(x) = (x^5 - \sqrt[5]{x})\left(x + \frac{1}{x}\right) = (x^5 - x^{1/5})(x + x^{-1})$

$$\boxed{f'(x) = \left(5x^4 - \frac{1}{5}x^{-4/5}\right)(x + x^{-1}) + (x^5 - x^{1/5})(1 - x^{-2})}$$

[3]

(b) $y = \dfrac{2\sin(x)}{x - \cos(x)}$

$$\boxed{y' = \frac{(x - \cos(x))(2\cos(x)) - 2\sin(x)(1 + \sin(x))}{(x - \cos(x))^2}}$$

[3]

Question 6 [4 points]: Find y'' if $y = \csc(x)$.

$y' = -\csc(x)\cot(x)$

$y'' = \csc(x)\cot(x) \cdot \cot(x) + (-\csc(x))(-\csc^2(x))$

$= \boxed{\csc(x)\cot^2(x) + \csc^3(x)}$

[4]

Question 7 [5 points]: At time t seconds, the position function for a particle moving along a straight line is $s(t) = t^3 - 6t^2 + 9t$ metres. What is the particle's velocity when the acceleration is zero?

$s'(t) = 3t^2 - 12t + 9$

$s''(t) = 6t - 12$

$s''(t) = 0$ at $t = 2$

$s'(2) = 3(2)^2 - 12(2) + 9 = \boxed{-3 \ \frac{m}{s}}$

[5]

Question 8 [5 points]: Find the x-coordinates of the points on the graph of $g(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 3$ at which tangent lines are parallel to the line $y = -4x - 7$.

Solve $g'(x) = -4$ for x:

$\left(\frac{1}{3}\right)3x^2 - \left(\frac{5}{2}\right)2x = -4$

$x^2 - 5x + 4 = 0$

$(x-4)(x-1) = 0$

$\boxed{x = 4, \quad x = 1}$

[5]