Question 1. [5]: Show that $u = 2x - 2xy$ is harmonic and find its harmonic conjugate.

$u_x = 2 - 2y \qquad u_y = -2x$

$u_{xx} = 0 \qquad u_{yy} = 0$

$\therefore u_{xx} + u_{yy} = 0$, so u is harmonic.

$$u_x = v_y \Rightarrow v = \int 2 - 2y \, dy = 2y - \frac{2y^2}{2} + g(x)$$

$$v_x = -u_y \Rightarrow g'(x) = -(-2x) = 2x$$

$$\Rightarrow g(x) = x^2 + C$$

$$\therefore v = 2y - y^2 + x^2 + C$$

Question 2. [5]: Without calculating any partial derivatives, explain why the function $\mathrm{Re}\left(\frac{\cos(z)}{e^z}\right)$ is harmonic in the whole complex plane.

$\cos(z)$ and e^z are entire functions,

and $e^z \neq 0$ for every $z \in \mathbb{C}$.

$\therefore \frac{\cos(z)}{e^z}$ is entire.

$\therefore \mathrm{Re}\left(\frac{\cos(z)}{e^z}\right)$ is harmonic since it is the real part of an analytic function.

Question 3. [5]: Determine the imaginary part of $\sin\left(\frac{\pi}{2} - i\right)$.

$$\sin\left(\frac{\pi}{2}-i\right) = \frac{1}{2i}\left[e^{i(\frac{\pi}{2}-i)} - e^{-i(\frac{\pi}{2}-i)}\right]$$

$$= \frac{1}{2i}\left[e^{i\frac{\pi}{2}+1} - e^{-i\frac{\pi}{2}-1}\right]$$

$$= \frac{1}{2i}\left[e^{i\frac{\pi}{2}}e^{+1} - e^{-i\frac{\pi}{2}}e^{-1}\right]$$

$$= \frac{1}{2i}\left[i \cdot e - (-i)e^{-1}\right]$$

$$= \frac{1}{2}\left(e+e^{-1}\right) \leftarrow \text{real !}$$

$$\therefore \ \operatorname{Im}\left(\sin\left(\frac{\pi}{2}-i\right)\right) = \boxed{0}$$

Question 4. [5]: Determine all solutions $z \in \mathbb{C}$ to

$$\cos z = i \sin z$$

$$\frac{e^{iz}+e^{-iz}}{2} = i\,\frac{e^{iz}-e^{-iz}}{2i}$$

$$\Rightarrow \quad e^{iz}+e^{-iz} = e^{iz}-e^{-iz}$$

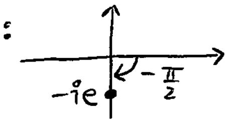
$$\Rightarrow \quad 2e^{-iz} = 0$$

$$\Rightarrow \quad e^{-iz} = 0$$

$\therefore$ no solutions since $e^{\omega} \neq 0$ for all $\omega \in \mathbb{C}$.

Question 5. [3]: Determine all values of $\log(-ie)$.

$\mathbb{C}$:



$$\log(-ie) = \mathrm{Log}|-ie| + i\,\mathrm{Arg}(-ie) + i2k\pi$$

$$= \mathrm{Log}|e| + i\left(\frac{-\pi}{2}\right) + i2k\pi$$

$$= 1 + i\left(2k - \frac{1}{2}\right)\pi, \quad k \in \mathbb{Z}.$$

Question 6. [3]: Solve for z:

$$\mathrm{Log}(z^2 + i) = i\frac{\pi}{2}$$

$$\Rightarrow e^{\mathrm{Log}(z^2+i)} = e^{i\frac{\pi}{2}}$$

$$z^2 + i = i$$

$$z^2 = 0$$

$$\boxed{z = 0}$$

Question 7. [4]: Let $f(z) = z^{1+z}$ where the complex power is defined using the principal branch of the logarithm. Evaluate and simplify $f(i)$.

$$f(i) = i^{1+i} = e^{(1+i)\mathrm{Log}(i)}$$

$$\mathrm{Log}(i) = \mathrm{Log}|i| + i\,\mathrm{Arg}(i)$$

$$= \cancel{\mathrm{Log}\,1}^{\,0} + i\frac{\pi}{2}$$

$$\therefore f(i) = e^{(1+i)i\frac{\pi}{2}} = e^{i\frac{\pi}{2} - \frac{\pi}{2}} = e^{i\frac{\pi}{2}} e^{-\frac{\pi}{2}} = ie^{-\frac{\pi}{2}}.$$

Question 8. [7]: Evaluate

$$I = \int_\Gamma (\bar{z})^2\, dz$$

where Γ is the circle with centre $z = 1$ and radius 1 traversed once in the positive direction.

$$z(t) = 1 + e^{it}, \quad 0 \le t \le 2\pi$$

$$z'(t) = ie^{it}$$

$$I = \int_0^{2\pi} \left(\overline{1+e^{it}}\right)^2 ie^{it}\, dt$$

$$= \int_0^{2\pi} (1 + e^{-it})^2\, ie^{it}\, dt$$

$$= \int_0^{2\pi} \left(1 + 2e^{-it} + e^{-2it}\right) ie^{it}\, dt$$

$$= i\int_0^{2\pi} \left(e^{it} + 2 + e^{-it}\right) dt$$

$$= i\left(\frac{e^{it}}{i}\Big|_0^{2\pi} + 2t\Big|_0^{2\pi} - \frac{e^{-it}}{i}\Big|_0^{2\pi} \right)$$

$$= \boxed{4\pi i}$$

Question 9. [3]: With reference to your answer to Question 8, does $f(z) = (\bar{z})^2$ have an antiderivative in any domain D containing the circle of centre $z = 1$ and radius 1? Explain.

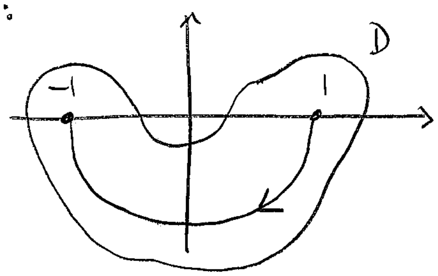
No. If f had an antiderivative in such a domain, then every loop integral in the domain would be zero. The solution to Question 8 shows there is at least one loop integral of $f(z) = (\bar{z})^2$ which does not evaluate to zero, so f can have no antiderivative in such a domain.

Question 10. [5]: Evaluate

$$\int_{\Gamma} \left(e^z - \frac{1}{z^2} \right) dz$$

where Γ is the lower half of the unit circle traversed in the negative (i.e. clockwise) direction.

$\mathbb{C}$:



$f(z) = e^z - \frac{1}{z^2}$ has antiderivative $F(z) = e^z + \frac{1}{z}$ in D.

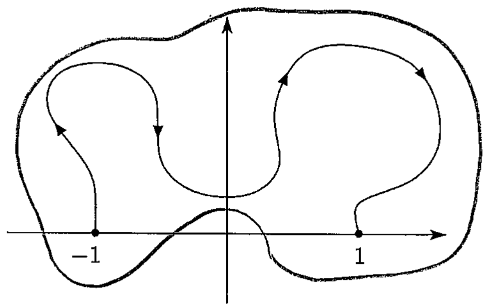
$$\therefore \int_{\Gamma} f(z)\,dz = F(-1) - F(1)$$

$$= \left[e^{-1} + \frac{1}{(-1)}\right] - \left[e^{1} + \frac{1}{1}\right]$$

$$= e^{-1} - e - 2.$$

Question 11. [5]:

Evaluate $\int_{\Gamma} \frac{1}{z}\, dz$ where Γ is the indicated contour from -1 to 1:



$f(z) = \frac{1}{z}$ has antiderivative

$F(z) = \mathcal{L}_{-\frac{\pi}{2}}(z) = \mathrm{Log}|z| + i \arg_{-\frac{\pi}{2}}(z)$ in D

$$\therefore \int_{\Gamma} \frac{1}{z}\, dz = F(1) - F(-1)$$

$$= \left[\cancel{\mathrm{Log}|1|}^{\,0} + i \arg_{-\frac{\pi}{2}}(1)\right] - \left[\cancel{\mathrm{Log}|-1|}^{\,0} + i \arg_{-\frac{\pi}{2}}(-1)\right]$$

$$= i \cdot 0 - i\pi$$

$$= -i\pi$$