Question 1. [5]: Show that if $|z| = 1$, but $z \neq 1$, then $\text{Re}\left(\frac{1}{1-z}\right) = \frac{1}{2}$.

Let $z = x+iy$ where $x^2+y^2=1$.

$$\text{Then } \text{Re}\left(\frac{1}{1-z}\right) = \text{Re}\left(\frac{1}{1-x-iy}\right)$$

$$= \text{Re}\left(\frac{1-x+iy}{(1-x-iy)(1-x+iy)}\right)$$

$$= \text{Re}\left(\frac{1-x}{(1-x)^2+y^2} + i\frac{y}{(1-x)^2+y^2}\right)$$

$$= \frac{1-x}{1-2x+\underbrace{x^2+y^2}_{=1}}$$

$$= \frac{\cancel{(1-x)}}{2\cancel{(1-x)}} = \frac{1}{2}.$$

Question 2. [5]: Let $a_1, a_2, \ldots, a_n$ be real constants. Show that if $\overline{z_0}$ is a root of

$$z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n = 0$$

then so is z_0.

We have $(\overline{z_0})^n + a_1(\overline{z_0})^{n-1} + a_2(\overline{z_0})^{n-2} + \cdots + a_n = 0$

$$\Rightarrow \overline{(\overline{z_0})^n + a_1(\overline{z_0})^{n-1} + a_2(\overline{z_0})^{n-2} + \cdots + a_n} = \overline{0}$$

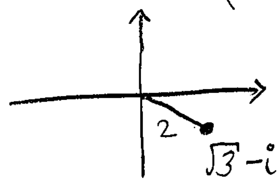
$$\Rightarrow z_0^n + a_1 z_0^{n-1} + a_2 z_0^{n-2} + \cdots + a_n = 0,$$

so z_0 is a root

(Here we have used the facts that
(i) $a \in \mathbb{R} \Rightarrow \bar{a} = a$,
(ii) $\overline{(\bar{z})^k} = (\bar{\bar{z}})^k = z^k$.)

Question 3. [3]: Determine $\arg\left((\sqrt{3}-i)^2\right)$.

$\mathbb{C}$:



$\because \quad \sqrt{3}-i = 2e^{-i\frac{\pi}{6}}$

$\therefore \quad (\sqrt{3}-i)^2 = \left(2e^{-i\frac{\pi}{6}}\right)^2 = 4e^{-i\frac{\pi}{3}}$

$\therefore \quad \arg(\sqrt{3}-i)^2 = -\frac{\pi}{3} + 2k\pi, \ k \in \mathbb{Z}.$

Question 4. [4]: Is $\text{Arg}(\overline{z}) = -\text{Arg}(z)$ for every $z \in \mathbb{C}$? Either prove the statement or find a $z \in \mathbb{C}$ for which the statement is not true.

Statement is false: consider $z = -1$, $\text{Arg}(z) = \pi$.
$\overline{z} = -1$, $\text{Arg}(\overline{z}) = \pi \neq -\text{Arg}(z)$.

Question 5. [3]: Show that $\overline{(e^z)} = e^{\overline{z}}$.

Let $z = x + iy$.

$e^z = e^x[\cos y + i\sin y]$

$\overline{(e^z)} = e^x[\cos y - i\sin y]$

$= e^x[\cos(-y) + i\sin(-y)]$

$= e^{x-iy} = e^{\overline{z}}.$

Question 6. [5]: Show that $f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$ maps the unit circle $|z| = 1$ to the interval $[-1, 1]$ on the real axis.

Suppose $|z| = 1$, so $x^2 + y^2 = 1$,

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$= \frac{1}{2}\left(x + iy + \frac{1}{x+iy}\right)$$

$$= \frac{1}{2}\left(x + iy + \frac{x - iy}{\cancel{x^2+y^2}}\right) \quad 1$$

$$= \frac{1}{2}\left(x + \cancel{iy} + x - \cancel{iy}\right)$$

$= x$, and $|z| = 1 \Rightarrow -1 \leq x \leq 1$.

Question 7. [5]: Let $\beta_1, \beta_2, \beta_3, \beta_4$ be the fourth roots of i. Determine the value of the product $\beta_1\beta_2\beta_3\beta_4$.

$$i = e^{i\frac{\pi}{2}}$$

$$\therefore i^{\frac{1}{4}} = e^{\frac{i\left(\frac{\pi}{2} + 2k\pi\right)}{4}}, \quad k = 0, 1, 2, 3$$

$$\therefore \beta_1 = e^{i\frac{\pi}{8}}, \quad \beta_2 = e^{i\frac{5\pi}{8}}, \quad \beta_3 = e^{i\frac{9\pi}{8}}, \quad \beta_4 = e^{i\frac{13\pi}{8}}$$

$$\therefore \beta_1\beta_2\beta_3\beta_4 = e^{i(1+5+9+13)\frac{\pi}{8}}$$

$$= e^{i\frac{28\pi}{8}}$$

$$= e^{i\frac{24\pi}{8}} e^{i\frac{4\pi}{8}}$$

$$= e^{i3\pi} e^{i\frac{\pi}{2}} = \boxed{-i}$$

Question 8. [5]: Show that $f(x+iy) = x^2 + y^2 + y - 2 + ix$ is nowhere analytic.

Here $u(x,y) = x^2 + y^2 + y - 2$

$v(x,y) = x$

$$\left.\begin{array}{l} u_x = 2x \;,\; v_y = 0 \\ u_y = 2y+1 \;,\; -v_x = -1 \end{array}\right\} \text{all first partial derivatives continuous on } \mathbb{R}^2.$$

$u_x = v_y \Rightarrow x = 0$

$u_y = -v_x \Rightarrow y = -1$

$\therefore$ f is differentiable at the single point $z_0 = -i$

$\therefore$ f is not analytic at any $z \neq z_0$ and not analytic on any neighbourhood of z_0, so f is nowhere analytic.

Question 9. [5]: Show that $f(x+iy) = 3x^2 + 2x - 3y^2 - 1 + i(6xy + 2y)$ is entire.

$u(x,y) = 3x^2 + 2x - 3y^2 - 1$

$v(x,y) = 6xy + 2y$

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial x} = 6x + 2 = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -6y = -\dfrac{\partial v}{\partial x}. \end{array}\right\} \text{all first partial derivatives continuous on } \mathbb{R}^2.$$

$\therefore$ f is differentiable at every $z \in \mathbb{C}$, so f is entire.

Question 10. [5]: Show that if $z_1 + z_2$ and $z_1\overline{z_2}$ are both real, then either $z_1 = -z_2$ or z_1 and z_2 are both real.

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$.

$z_1 + z_2$ real $\Rightarrow$ $(x_1 + x_2) + i(y_1 + y_2)$ real

$z_1\overline{z_2}$ real $\Rightarrow$ $(x_1 + iy_1)(x_2 - iy_2)$ real $\Rightarrow (x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)$ real.

Suppose z_1 & z_2 are not both real; we must show $z_1 = -z_2$.

z_1 & z_2 not both real $\Rightarrow$ $y_1 y_2 \neq 0$; without loss of generality suppose $y_2 \neq 0$.

Now $z_1 + z_2$ real $\Rightarrow$ $y_1 + y_2 = 0$ $\Rightarrow$ $y_1 = -y_2$.

$z_1\overline{z_2}$ real $\Rightarrow$ $x_2y_1 - x_1y_2 = 0$ $\Rightarrow$ $-x_2y_2 - x_1y_2 = 0$

$\Rightarrow$ $-x_2 = x_1$ upon dividing by $y_2 \neq 0$.

So $y_1 = -y_2$ & $x_1 = -x_2$,

i.e. $z_1 = x_1 + iy_1 = -x_2 - iy_2 = -z_2$ as required.

Question 11. [5]: Show that if both $f(z)$ and $\overline{f(z)}$ are analytic in a domain D then $f(z)$ must be constant in D.

Let $f(z) = u(x,y) + iv(x,y)$.

f analytic $\Rightarrow$ $u_x = v_y$ ① throughout D,
$u_y = -v_x$ ②

$\overline{f(z)} = u(x,y) - iv(x,y)$

$\overline{f}$ analytic $\Rightarrow$ $u_x = -v_y$ ③ throughout D.
$u_y = v_x$ ④

① & ③ $\Rightarrow$ $v_y = 0$ $\Rightarrow$ $u_x = 0$ throughout D.

② & ④ $\Rightarrow$ $v_x = 0$ $\Rightarrow$ $u_y = 0$ throughout D.

$\therefore$ $u_x = u_y = 0$ on D, so $u(x,y)$ is constant on D;
$v_x = v_y = 0$ on D, so $v(x,y)$ is constant on D.
$\therefore$ $f(z) = u(x,y) + iv(x,y)$ is constant on D.