1. $$\frac{5+5i}{(1+3i)\left(\frac{1}{2}-\frac{i}{2}\right)} = \frac{10+10i}{(1+3i)(1-i)}$$

$$= \frac{10\,(1+i)}{1+3i-i+3}$$

$$= \frac{10(1+i)}{4+2i} \cdot \frac{4-2i}{4-2i}$$

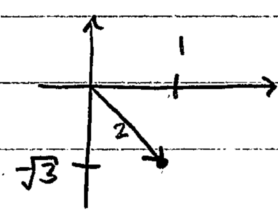
$$= \frac{10\,(4+4i-2i+2)}{4^2+2^2}$$

$$= \frac{\cancel{10}(6+2i)}{\cancel{20}\ 2}$$

$$= 3+i$$

---

2. $1-\sqrt{3}\,i$:



$$\therefore\ 1-\sqrt{3}\,i = 2e^{-i\frac{\pi}{3}}.$$

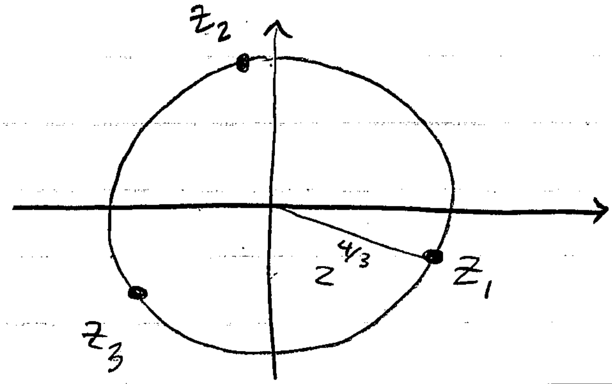
$$\therefore\ z = 8(1-\sqrt{3}) = 2^4 e^{-i\pi/3}.$$

$$\therefore\ z^{1/3} = \left(2^4\right)^{\frac{1}{3}} e^{i\frac{\left[-\pi/3+2k\pi\right]}{3}}, \quad k=0,1,2$$

$$= 2^{\frac{4}{3}} e^{i\left[\frac{-\pi}{9}+\frac{6k\pi}{9}\right]}, \quad k=0,1,2$$

$$= 2^{\frac{4}{3}} e^{-i\frac{\pi}{9}},\ 2^{\frac{4}{3}} e^{i\frac{5\pi}{9}},\ 2^{\frac{4}{3}} e^{i\frac{11\pi}{9}}$$

$$= z_1,\ z_2,\ z_3 \quad \text{(say)} \longrightarrow.$$



3. $z = 6e^{i\pi/3} = 6\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$

$= 6\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right].$

$= 3 + i\,3\sqrt{3}$

$\therefore\ e^{iz} = e^{i3 - 3\sqrt{3}} = e^{i3}e^{-3\sqrt{3}}$

$\therefore\ |e^{iz}| = \underbrace{\cancel{|e^{i3}|}}_{1}\,|e^{-3\sqrt{3}}| = e^{-3\sqrt{3}}.$

4. $f(z) = |\bar{z} - i|^2$

$= |\overline{x+iy} - i|^2$

$= |x - iy - i|^2$

$= |x - i(1+y)|^2$

$= x^2 + (1+y)^2$

$\therefore\ u(x,y) = x^2 + (1+y)^2,\quad v(x,y) = 0.$

$u_x = 2x,\quad v_y = 0$

$u_y = 2(1+y),\quad -v_x = 0.$ $\longrightarrow$

$$u_x = v_y \Rightarrow x = 0$$
$$u_y = -v_x \Rightarrow y = -1.$$

$\therefore$ C.R. equations satisfied at $(x,y) = (0,-1)$ only, but not in any neighbourhood of this point, so $f(z)$ is nowhere analytic.

---

5. $u(x,y) = 3x^2y - y^3 + x + 4xy$

$$u_x = 6xy + 1 + 4y, \quad u_y = 3x^2 - 3y^2 + 4x$$

$$u_x = v_y \Rightarrow v = \int 6xy + 1 + 4y \, dy$$

$$= \frac{6xy^2}{2} + y + \frac{4y^2}{2} + g(x)$$
$$= 3xy^2 + y + 2y^2 + g(x).$$

$$u_y = -v_x \Rightarrow -\frac{\partial}{\partial x}\left[3xy^2 + y + 2y^2 + g(x)\right] = u_y$$

$$\Rightarrow -3y^2 - g'(x) = 3x^2 - 3y^2 + 4x$$

$$\therefore -g'(x) = 3x^2 + 4x$$
$$g'(x) = -(3x^2 + 4x)$$
$$g(x) = -\left(\frac{3x^3}{3} + \frac{4x^2}{2}\right) + C$$
$$= -(x^3 + 2x^2) + C.$$

$$\therefore v(x,y) = 3xy^2 + y + 2y^2 - x^3 - 2x^2 + C.$$

6. If $u(x,y) = \mathrm{Re}(f(z))$ where f is entire, then $u_{xx} + u_{yy} \equiv 0$.

$u(x,y) = xy^2$

$u_x = y^2 \qquad u_y = 2xy$

$u_{xx} = 0 \qquad u_{yy} = 2x$

Since $u_{xx} + u_{yy} = 2x \not\equiv 0$, $u(x,y)$ is not the real part of an entire function.

---

7. $$\mathrm{Log}(i) = \mathrm{Log}\left(1 \cdot e^{i\pi/2}\right) = \cancel{\mathrm{Log}|1|}^{\,0} + i\frac{\pi}{2} = i\frac{\pi}{2}.$$

$$i^{(i\pi/2)} = e^{(i\pi/2)\log(i)}$$

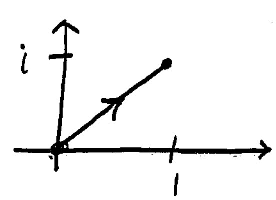
$$= e^{(i\pi/2)\left[\mathrm{Log}(i) + i2\pi k\right]}, \quad k \in \mathbb{Z}$$

$$= e^{(i\pi/2)\left[i\frac{\pi}{2} + i2\pi k\right]}, \quad k \in \mathbb{Z}$$

$$= e^{-\pi^2/4 - \pi^2 k}$$

$$= e^{-(k+\frac{1}{4})\pi^2}, \quad k \in \mathbb{Z}.$$

8(a). $\int_\Gamma \mathrm{Re}(z)\,dz$; Γ :



$z(t) = (1+i)t = t + it, \quad 0 \le t \le 1$

$z'(t) = (1+i)$.

$\longrightarrow$

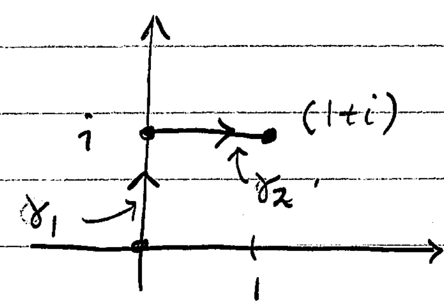
$$\therefore \int_\Gamma \mathrm{Re}(z)\,dz = \int_0^1 \mathrm{Re}(t+it)\,(1+i)\,dt$$

$$= (1+i)\int_0^1 t\,dt$$

$$= (1+i)\left.\frac{t^2}{2}\right|_0^1$$

$$= \frac{1+i}{2}.$$

8(b): $\displaystyle\int_\Gamma \mathrm{Re}(z)\,dz$, Γ:



γ_1: $z(t) = it$, $0 \le t \le 1$.
$z'(t) = i$

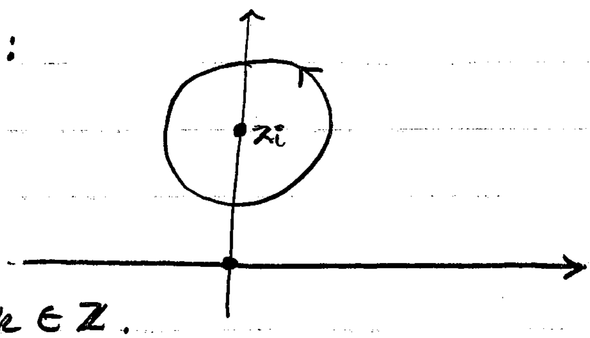
$$\int_{\gamma_1} \mathrm{Re}(z)\,dz = \int_0^1 0 \cdot i\,dt = 0.$$

γ_2: $z(t) = i + t$, $0 \le t \le 1$
$z'(t) = 1$

$$\int_{\gamma_2} \mathrm{Re}(z)\,dz = \int_0^1 t \cdot 1\,dt = \left.\frac{t^2}{2}\right|_0^1 = \frac{1}{2}.$$

$$\therefore \int_\Gamma \mathrm{Re}(z)\,dz = \int_{\gamma_1} + \int_{\gamma_2} = 0 + \frac{1}{2} = \frac{1}{2}.$$

9. $\displaystyle\int_C \frac{\cos z}{e^z - 1}\, dz$, C:



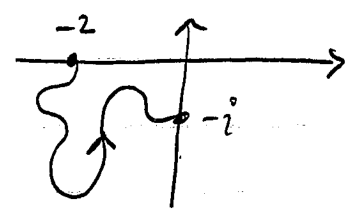
$e^z - 1 = 0$ at $z = 2k\pi i$, $k \in \mathbb{Z}$.

$\therefore$ $\dfrac{\cos z}{e^z-1}$ is analytic inside and on C,

so by Cauchy's Integral Formula, $\displaystyle\int_C \frac{\cos z}{e^z-1}\, dz = 0$.

---

10. $\displaystyle\int_\Gamma \frac{1}{z}\, dz$, Γ:



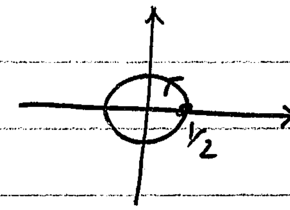
$\mathcal{L}_0(z) = \mathrm{Log}(z) + i\arg_0(z)$ has derivative $\frac{1}{z}$ in a domain D containing Γ.

$$\therefore \int_\Gamma \frac{1}{z}\, dz = \mathcal{L}_0(-i) - \mathcal{L}_0(-2)$$

$$= \Big[\underbrace{\cancel{\mathrm{Log}|-i|}}_{0} + i\frac{3\pi}{2}\Big] - \Big[\mathrm{Log}|-2| + i\pi\Big]$$

$$= -\mathrm{Log}|2| + i\frac{\pi}{2}.$$

11(a). $I_1 = \int_C \frac{z^3}{(z+i)(z+2)^2}\, dz$ where C:

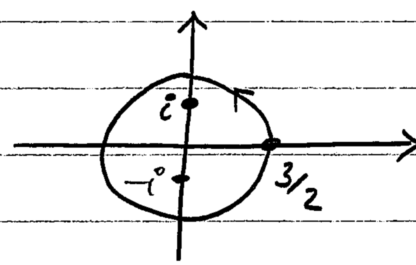


$\frac{z^3}{(z+i)(z+2)^2}$ is analytic inside and on C, so

$I_1 = \int_C \frac{z^3}{(z+i)(z+2)^2}\, dz = 0$ by Cauchy's Integral Thm.

11(b). $I_2 = \int_C \frac{z^3}{(z+i)(z+2)^2}\, dz = \int_C \frac{\left[z^3/(z+2)^2\right]}{z-(-i)}\, dz$

where C:

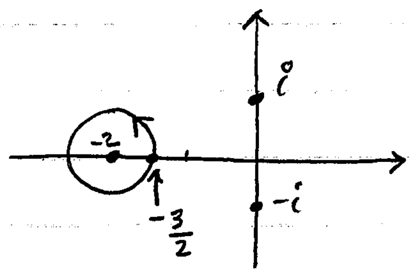


$g(z) = \frac{z^3}{(z+2)^2}$ is analytic inside and on C, so by Cauchy's Integral Formula,

$$\begin{aligned}
I_2 &= 2\pi i\, g(-i) \\
&= \frac{2\pi i \, (-i)^3}{(2-i)^2} \\
&= \frac{-2\pi i}{(2-i)^2} \\
&= \frac{-2\pi i}{3-4i} \cdot \frac{3+4i}{3+4i} \\
&= \frac{-6\pi - 8\pi i}{25}
\end{aligned}$$

11(c): $$I_3 = \int_C \frac{z^3}{(z+i)(z+2)^2}\,dz = \int_C \frac{\left[z^3/(z+i)\right]}{(z-(-2))^2}\,dz$$

where C:



$g(z) = \dfrac{z^3}{z+i}$ is analytic inside and on C, so by the generalized Cauchy Integral Formula,

$$I_3 = 2\pi i\, g'(-2)$$

$$= 2\pi i \left[\frac{(z+i)\,3z^2 - z^3}{(z+i)^2}\right]_{z=-2}$$

$$= 2\pi i \left[\frac{(-2+i)\cdot 3\cdot(-2)^2 - (-2)^3}{(-2+i)^2}\right]$$

$$= 2\pi i \left[\frac{-24+12i+8}{(-2+i)^2}\right]$$

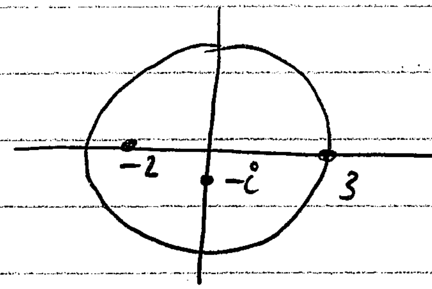
$$= \frac{-24\pi - 32\pi i}{(-2+i)^2}$$

$$= \frac{-8\pi\,[3+4i]}{(-2+i)^2}$$

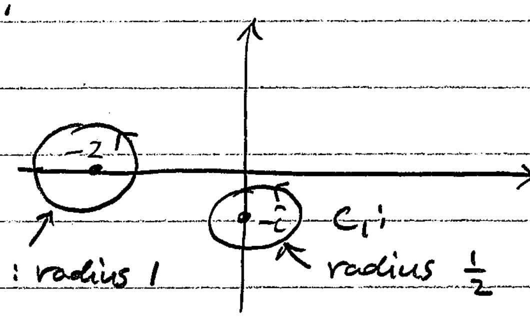
$$= \frac{-8\pi\,[3+4i]}{3-4i} \cdot \frac{3+4i}{3+4i}$$

$$= \frac{-8\pi\,[-7+24i]}{25}$$

11(d): $I_4 = \int_C \frac{z^3}{(z+i)(z+2)^2}\,dz$, C :



Deform C into C_1 , C_2 :

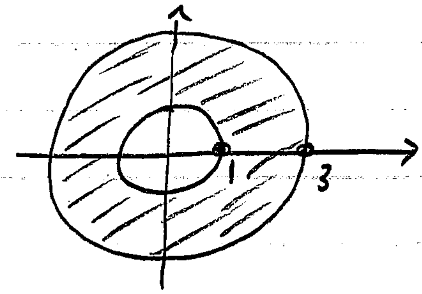


$$\therefore \quad I_4 = I_2 + I_3 = \frac{-6\pi - 8\pi i}{25} + \frac{-8\pi[-7+24i]}{25}$$

$$= \frac{50\pi - 200\pi i}{25}$$

$$= 2\pi - 8\pi i$$

12. $f(z) = \dfrac{1}{(z+1)(z+3)}$,



$$f(z) = \frac{(1/2)}{z+1} + \frac{(-1/2)}{z+3} = \frac{1}{2(z+1)} - \frac{1}{2(z+3)} .$$

We require a series for $\dfrac{1}{2(z+1)}$ valid for $|z|>1$,

so $\dfrac{1}{|z|} < 1$: $\dfrac{1}{2(z+1)} = \dfrac{1}{2z}\,\dfrac{1}{1-(-\frac{1}{z})}$

$$= \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \cdots\right]$$

$$= \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} + \cdots$$

$$= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{2z^j}$$

Series for $\dfrac{1}{2(z+3)}$ must be valid for $|z|<3$,

so $\left|\dfrac{z}{3}\right| < 1$: $\dfrac{1}{2(z+3)} = \dfrac{1}{6}\,\dfrac{1}{1-(-\frac{z}{3})}$

$$= \frac{1}{6}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right]$$

$$= \frac{1}{6} - \frac{z}{6\cdot 3} + \frac{z^2}{6\cdot 3^2} - \cdots$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j z^j}{6\cdot 3^j}$$

$\therefore$ On $1 < |z| < 3$,

$$f(z) = \left(\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{2z^j}\right) - \left(\sum_{j=0}^{\infty} \frac{(-1)^j z^j}{6\cdot 3^j}\right) .$$

13. $\displaystyle \lim_{z\to -1} \frac{1+\cos(\pi z)}{(z^2-1)^2} \sim \frac{0}{0}$

$$\overset{H}{=} \lim_{z\to -1} \frac{-\pi\sin(\pi z)}{2(z^2-1)(2z)} \sim \frac{0}{0}$$

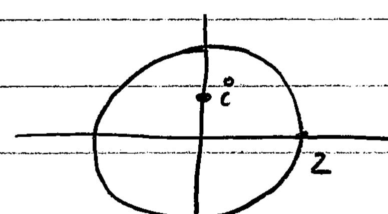
$$\overset{H}{=} \lim_{z\to -1} \frac{-\pi^2\cos(\pi z)}{(2z)(4z)+(z^2-1)(4)}$$

$$= \frac{\pi^2}{8}$$

} Since $\displaystyle\lim_{z\to -1} f(z)$ exists,

$f(z)$ has a removable singularity at $z=-1$.

14(a) $\displaystyle I = \int_{|z|=2} \frac{z^3+2z}{z-i}\, dz$:



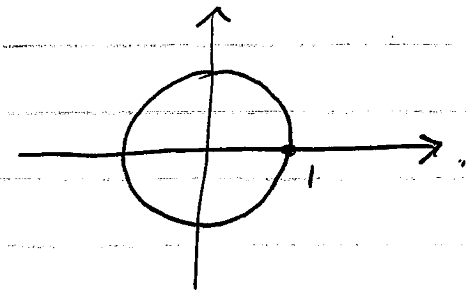
$f(z) = \dfrac{z^3+2z}{z-i} = \dfrac{g(z)}{z-i}$ where $g(z)$ is

analytic and $g(i) \neq 0$, so $z=i$ is a simple pole.

$\displaystyle \text{Res}(f;i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i} z^3+2z = i^3+2i = i$

$\therefore$ By the residue thm, $I = 2\pi i(i) = -2\pi$.

14(b) $I = \int_{|z|=1} z^2 e^{\frac{1}{z}}\, dz$:



Hee $f(z)$ has an essential singularity at $z=0$.
The Laurent series for $f(z)$ about $z=0$ is

$$z^2\left[1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots\right]$$

$$= z^2 + z + \frac{1}{2!} + \frac{1}{3!\,z} + \frac{1}{4!\,z^2} + \cdots$$

$\therefore$ $\text{Res}(f; 0) = \frac{1}{3!} = \frac{1}{6}$.

$\therefore$ By the Residue Thm,

$$I = 2\pi i\left(\frac{1}{6}\right) = \frac{i\pi}{3}.$$