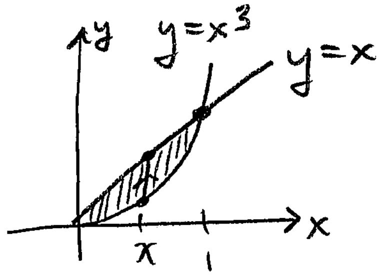
**Question 1. [10]:**

**(a)[5]** Compute $\iint_D (x^2+2y)\,dA$ where $D$ is the region in the first quadrant that is bounded between the curves $y=x$ and $y=x^3$.




$$\iint_D x^2+2y\,dA = \int_{x=0}^{1}\int_{y=x^3}^{x}(x^2+2y)\,dy\,dx$$

$$=\int_0^1 \left[x^2y+y^2\right]_{x^3}^{x}dx$$

$$=\int_0^1 x^3+x^2-x^5-x^6\,dx$$

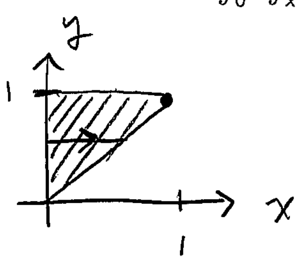
$$=\left[\frac{x^4}{4}+\frac{x^3}{3}-\frac{x^6}{6}-\frac{x^7}{7}\right]_0^1$$

$$=\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{7}$$

$$=\frac{21+28-14-12}{84}$$

$$=\boxed{\frac{23}{84}}$$

**(b)[5]** Evaluate $\int_0^1\int_x^1 e^{x/y}\,dy\,dx$ by first changing the order of integration.



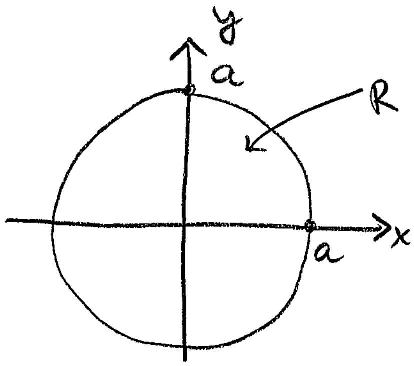

$$\int_0^1\int_x^1 e^{\frac{x}{y}}dy\,dx = \int_{y=0}^{1}\int_{x=0}^{y} e^{\frac{x}{y}}dx\,dy$$

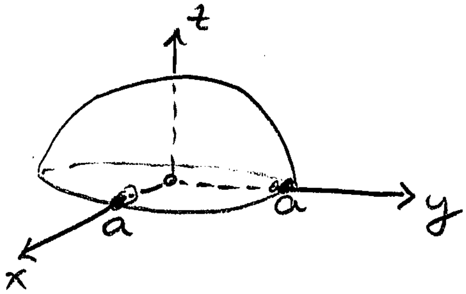
$$=\int_0^1 y\left[e^{x/y}\right]_{x=0}^{y}dy$$

$$=\int_0^1 y\left[e-1\right]dy$$

$$=(e-1)\left.\frac{y^2}{2}\right|_0^1 = \boxed{\frac{e-1}{2}}$$

**Question 2. [10]:** The upper hemisphere of a sphere of radius $a$ is given by $z = \sqrt{a^2 - x^2 - y^2}$. Determine the volume of the hemisphere using polar coordinates.
(You may know the answer directly thanks to a well-known formula; you are required here to show how to get that answer.)




$x = r\cos\theta$
$y = r\sin\theta$

$$\therefore \quad V = \iint_R \sqrt{a^2 - x^2 - y^2}\, dA$$

$$= \int_0^{2\pi}\int_0^a \sqrt{a^2 - r^2\cos^2\theta - r^2\sin^2\theta}\; r\, dr\, d\theta$$

$$= \left(\tfrac{-1}{2}\right)\int_0^{2\pi}\int_0^a \underbrace{(a^2 - r^2)}_{u = a^2 - r^2,\ du = -2r\,dr}{}^{\frac{1}{2}} (-2r)\, dr\, d\theta$$

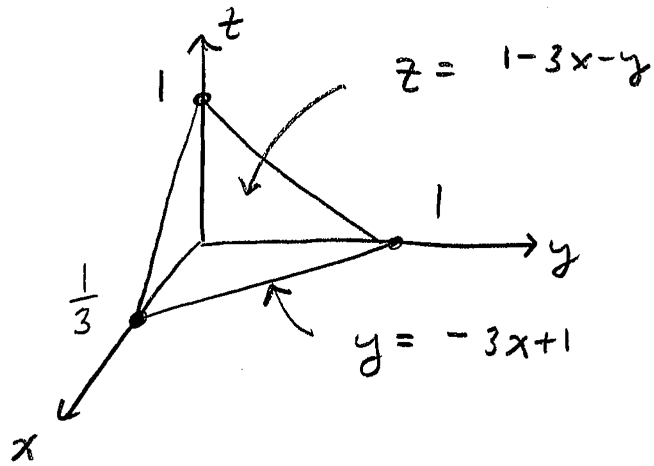
$$= \frac{-1}{2}\int_0^{2\pi} \left[a^2 - r^2\right]^{\frac{3}{2}} \cdot \left(\frac{2}{3}\right)\Big|_0^a \, d\theta$$

$$= \left(\frac{-1}{\cancel{2}}\right)\left(\frac{\cancel{2}}{3}\right)\int_0^{2\pi} -(a^2)^{\frac{3}{2}}\, d\theta$$

$$= \frac{1}{3}a^3\left[\theta\right]_0^{2\pi}$$

$$= \boxed{\frac{2\pi}{3}a^3}$$

**Question 3. [10]:** Evaluate $\iiint_E y\, dA$ where $E$ is the solid region in the first octant that lies under the plane $3x + y + z = 1$.




$$\iiint_E y\, dA = \int_{x=0}^{1/3} \int_{y=0}^{-3x+1} \int_{z=0}^{1-3x-y} y\, dz\, dy\, dx$$

$$= \int_0^{1/3} \int_0^{-3x+1} y\big[z\big]_0^{1-3x-y}\, dy\, dx$$

$$= \int_0^{1/3} \int_0^{-3x+1} y(1-3x-y)\, dy\, dx$$

$$= \int_0^{1/3} \left[\frac{(1-3x)}{2} y^2 - \frac{y^3}{3}\right]_0^{-3x+1} dx$$

$$= \int_0^{1/3} \frac{(1-3x)^3}{2} - \frac{(1-3x)^3}{3}\, dx$$

$$= \frac{1}{6} \cdot \frac{-1}{12} \left[(1-3x)^4\right]_0^{1/3}$$

$$= \frac{-1}{72}\left[0-1\right] = \boxed{\frac{1}{72}}$$

**Question 4. [10]:** The sphere $B$ of radius $a$ has equation $x^2+y^2+z^2=a^2$ and volume $V(B)=4\pi a^3/3$. The average distance from a point in the sphere to its centre is defined to be

$$\rho_{\text{avg}} = \frac{1}{V(B)} \iiint_B \sqrt{x^2+y^2+z^2}\, dV$$

Compute $\rho_{\text{avg}}$ .

Using spherical coordinates:

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

$$\iiint_B \sqrt{x^2+y^2+z^2}\, dV = \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{a} \rho \cdot \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi} \int_0^{2\pi} \left[\frac{\rho^4}{4}\right]_0^a \sin\phi \, d\theta\, d\phi$$

$$= \frac{a^4}{4} \int_0^{\pi} [\theta]_0^{2\pi} \sin\phi \, d\phi$$

$$= \frac{2\pi a^4}{4} \cdot \left[-\cos\phi\right]_0^{\pi}$$

$$= \frac{2\pi}{4} a^4 \cdot (1+1)$$

$$= \pi a^4$$

$$\therefore \rho_{\text{avg}} = \frac{1}{V(B)} \pi a^4 = \frac{\cancel{\pi} a^4}{\frac{4}{3}\cancel{\pi} a^3} = \boxed{\frac{3}{4} a}$$

**Question 5 [10]:** For this question we will use a transformation to evaluate $\iint_R (x+y)e^{x^2-y^2}\,dA$ where $R$ is the rectangle with vertices $(0,0)$, $(3/2,3/2)$, $(1,-1)$, $(5/2,1/2)$ .

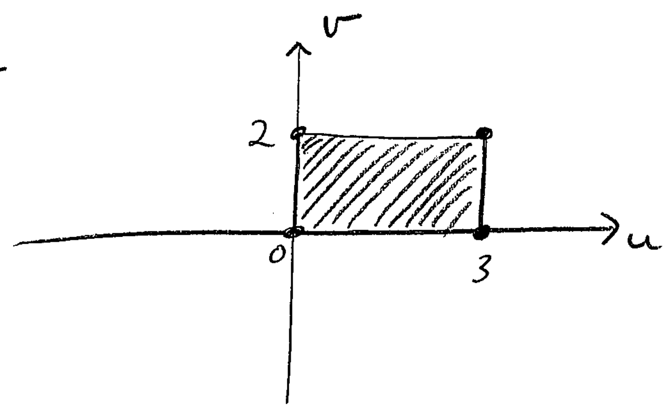
**(a)[2]** The transformation is $u = x+y$, $v = x-y$. Determine $\frac{\partial(x,y)}{\partial(u,v)}$, the Jacobian of the transformation.

$$\left.\begin{aligned} u &= x+y \\ v &= x-y \end{aligned}\right\} \quad \left.\begin{aligned} u+v &= 2x \\ u-v &= 2y \end{aligned}\right\} \quad \begin{aligned} x &= \tfrac{1}{2}(u+v) \\ y &= \tfrac{1}{2}(u-v) \end{aligned}$$

$$\therefore \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \boxed{-\frac{1}{2}}$$

**(b)[3]** Determine the region in the $uv$-plane that maps onto $R$ in the $xy$-plane.

| $(x,y)$ | $\longleftrightarrow$ $(u=x+y,\ v=x-y)$ |
|---|---|
| $(0,0)$ | $(0,0)$ |
| $(\frac{3}{2},\frac{3}{2})$ | $(3,0)$ |
| $(1,-1)$ | $(0,2)$ |
| $(\frac{5}{2},\frac{1}{2})$ | $(3,2)$ |




**(c)[5]** Use parts (a) and (b) to evaluate $\iint_R (x+y)e^{x^2-y^2}\,dA$ .

$$\iint_R (x+y)\,e^{x^2-y^2}\,dA = \int_0^3\int_0^2 u\,e^{uv}\left|-\tfrac{1}{2}\right|dv\,du$$

$$= \frac{1}{2}\int_0^3 \cancel{u}\left[\frac{e^{uv}}{\cancel{u}}\right]_0^2 du$$

$$= \frac{1}{2}\int_0^3 (e^{2u}-1)\,du$$

$$= \frac{1}{2}\left[\frac{e^{2u}}{2}-u\right]_0^3 = \frac{1}{2}\left[\frac{e^6}{2}-3-\frac{1}{2}\right] = \boxed{\frac{e^6-7}{4}}$$