Question 1. [10]:

(a)[5] Determine the direction derivative of $f(x,y,z) = x^2y + x\sqrt{1+z}$ at the point $(1,2,3)$ in the direction of $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 2,1,-2\rangle}{\sqrt{2^2+1^2+(-2)^2}} = \frac{1}{3}\langle 2,1,-2\rangle.$$

$$\nabla f(1,2,3) = \left\langle 2xy + \sqrt{1+z},\ x^2,\ \frac{x}{2}(1+z)^{-\frac{1}{2}} \right\rangle\Big|_{(1,2,3)}$$

$$= \left\langle (2)(1)(2) + \sqrt{1+3},\ 1^2,\ \frac{1}{2}(1+3)^{-\frac{1}{2}} \right\rangle$$

$$= \left\langle 6, 1, \frac{1}{4} \right\rangle$$

$$\therefore\ D_{\vec{u}} f(1,2,3) = \left\langle 6, 1, \frac{1}{4}\right\rangle \cdot \frac{\langle 2,1,-2\rangle}{3}$$

$$= \boxed{\frac{25}{6}}$$

(b)[5] Determine an equation of the tangent plane to the surface

$$\sin(xyz) = x + 2y + 3z$$

at the point $(2,-1,0)$.

Let $f(x,y,z) = \sin(xyz) - x - 2y - 3z.$

Equation of tangent plane is $\nabla f(2,-1,0) \cdot \langle x-2, y+1, z\rangle = 0$

$$\nabla f = \langle yz\cos(xyz) - 1,\ xz\cos(xyz) - 2,\ xy\cos(xyz) - 3\rangle$$

$$\therefore\ \nabla f(2,-1,0) = \langle -1, -2, -5\rangle$$

$\therefore$ Equation of tangent plane is

$$\langle -1,-2,-5\rangle \cdot \langle x-2, y+1, z\rangle = 0$$

$$\Rightarrow\ -(x-2) - 2(y+1) - 5z = 0$$

or $\boxed{x + 2y + 5z = 0}$

Question 2. [10]: Determine all critical points of $f(x, y) = x(x^2 + xy + y^2 - 9)$ and classify each as a corresponding to a local maximum, a local minimum, or a saddle point.

$$f(x,y) = x^3 + x^2y + xy^2 - 9x.$$

$$f_x = 3x^2 + 2xy + y^2 - 9, \quad f_{xx} = 6x + 2y$$

$$f_y = x^2 + 2xy, \quad f_{yy} = 2x$$

$$f_{xy} = 2x + 2y$$

Solving: ① $f_x = 3x^2 + 2xy + y^2 - 9 = 0$ ② $\Rightarrow x = 0$ or $x = -2y$

② $f_y = x(x+2y) = 0$

If $x=0$, ① becomes $y^2 - 9 = 0$, so $y = \pm 3$ giving C.P.s $(0,3)$, $(0,-3)$.

If $x = -2y$, ① becomes $12y^2 - 4y^2 + y^2 - 9 = 0$

$9y^2 - 9 = 0$

$y = \pm 1$,

giving C.P.s $(-2,1)$, $(2,-1)$.

C.P.	$D = (6x+2y)(2x) - (2x+2y)^2$	$f_{xx} = 6x+2y$	Classification
$(0,3)$	-36	—	saddle point
$(0,-3)$	-36	—	saddle point
$(-2,1)$	36	-10	local max.
$(2,-1)$	36	10	local min.

Question 3. [10]: Find three positive numbers whose sum is 100 and whose product is a maximum. You may use any method you like, but be sure to justify that your three numbers do indeed correspond to the absolute maximum.

Maximize $P = xyz$

Subject to $x+y+z = 100 \quad (x>0, y>0, z>0)$

So $z = 100 - x - y$

$\therefore \; P = xy(100-x-y) = 100xy - x^2y - xy^2$

① $P_x = 100y - 2xy - y^2 = y(100-2x-y) = 0$

② $P_y = 100x - 2xy - x^2 = x(100-2y-x) = 0$

Since $x>0, y>0$, ① & ② $\Rightarrow \begin{cases} 100-2x-y = 0 \\ 100-2y-x = 0 \end{cases}$

① − ② $\Rightarrow -2x+2y-y+x=0$

$\Rightarrow x = y$

① $\Rightarrow 100-2x-x=0 \Rightarrow x = \frac{100}{3}$

$\Rightarrow y = \frac{100}{3}$.

$D = P_{xx}P_{yy} - (P_{xy})^2 = (-2y)(-2x) - (100-2x-2y)^2$

$D(\frac{100}{3}, \frac{100}{3}) = \frac{40\,000}{9} - \frac{1000}{9} > 0$, and $P_{xx}(\frac{100}{3}, \frac{100}{3}) < 0$,

so $(x,y) = (\frac{100}{3}, \frac{100}{3})$ corresponds to a local max.

P clearly has an absolute max, which is also a local max, so $(\frac{100}{3}, \frac{100}{3})$ must correspond to the absolute max.

$\therefore \; x = \frac{100}{3}, \; y = \frac{100}{3}, \; z = 100-x-y = \frac{100}{3}$ are the required numbers.

Question 4. [10]: Use the method of Lagrange multipliers to find the absolute maximum value of

$$f(x, y) = 2x^2 + 3y^2 - 4x - 5$$

on the circle $x^2 + y^2 = 16$.

Maximize $f(x,y) = 2x^2+3y^2-4x-5$
subject to $g(x,y) = x^2+y^2=16$,

$$\begin{matrix} \nabla f = \lambda \nabla g \\ g(x,y) = 16 \end{matrix} \Rightarrow \left\{ \begin{array}{ll} ① & 4x-4 = \lambda \cdot 2x \\ ② & 6y = \lambda \cdot 2y \\ ③ & x^2+y^2=16. \end{array} \right\}$$

② $\Rightarrow$ $2y(\lambda-3)=0 \Rightarrow y=0,\ \lambda=3.$

- If $y=0$, ③ $\Rightarrow x = 4, -4$ giving points $(4,0), (-4,0)$.

- If $\lambda = 3$, ① $\Rightarrow 4x-4=6x$
$\Rightarrow 2(x+2)=0$
$\Rightarrow x=-2$,
so by ③ $y = \pm\sqrt{16-4} = \pm 2\sqrt{3}$,
giving points $(-2, 2\sqrt{3}), (-2, -2\sqrt{3})$.

Now test these points:

Point	$f(x,y) = 2x^2+3y^2-4x-5$
$(4,0)$	11
$(-4,0)$	43
$(-2, 2\sqrt{3})$	47
$(-2,-2\sqrt{3})$	47

$\therefore$ the absolute max. of $f(x,y)$ on $x^2+y^2=16$ is 47.

Question 5 [10]:

(a)[5] Evaluate

$$\int_1^2 \int_0^1 (x+y)^{-2}\,dx\,dy$$

$$= \int_1^2 \left[\frac{-1}{x+y}\right]_{x=0}^{x=1} dy$$

$$= \int_1^2 \frac{-1}{1+y} + \frac{1}{y}\,dy$$

$$= \left[-\ln|1+y| + \ln|y|\right]_1^2$$

$$= (-\ln 3 + \ln 2) - (-\ln 2 + \cancel{\ln 1}^{0})$$

$$= \boxed{2\ln 2 - \ln 3}$$

(b)[5] Determine the volume of the solid in the first octant which is enclosed by the surface $z = 1 + e^x \sin(y)$ and the planes $x = 1$, $y = \pi$.

$$V = \int_{x=0}^{1}\int_{y=0}^{\pi} (1 + e^x \sin y)\,dy\,dx$$

$$= \int_0^1 \left[y - e^x \cos y\right]_0^{\pi} dx$$

$$= \int_0^1 (\pi + e^x) - (0 - e^x)\,dx$$

$$= \int_0^1 \pi + 2e^x\,dx$$

$$= \left[\pi x + 2e^x\right]_0^1$$

$$= (\pi + 2e) - (0 + 2)$$

$$= \boxed{\pi + 2e - 2}$$