Question 1. [10]:

(a)[5] Determine the unit tangent vector $\mathbf{T}$ to $\mathbf{r}(t) = \cos(t)\,\mathbf{i} + 3t\,\mathbf{j} + 2\sin(2t)\,\mathbf{k}$ at the point where $t = 0$.

$$\vec{r}\,'(0) = -\sin(t)\,\hat{\imath} + 3\,\hat{\jmath} + 4\cos(2t)\,\hat{k}\,\Big|_{t=0}$$

$$= 3\,\hat{\jmath} + 4\,\hat{k}\,.$$

$$\therefore \vec{T} = \frac{\vec{r}\,'(0)}{|\vec{r}\,'(0)|} = \frac{3\hat{\jmath} + 4\hat{k}}{\sqrt{3^2+4^2}} = \boxed{\frac{3}{5}\hat{\jmath} + \frac{4}{5}\hat{k}}$$

(b)[5] Determine the position vector $\mathbf{r}(t)$ of a particle that has acceleration

$$\mathbf{a}(t) = t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}$$

where the intial velocity is $\mathbf{v}(0) = \mathbf{k}$ and the initial position is $\mathbf{r}(0) = \mathbf{j} + \mathbf{k}$.

$$\vec{v}(t) = \frac{t^2}{2}\hat{\imath} + e^t\hat{\jmath} - e^{-t}\hat{k} + \vec{C}_1$$

$$\vec{v}(0) = \hat{k} \Rightarrow \hat{\jmath} - \hat{k} + \vec{C}_1 = \hat{k}$$

$$\Rightarrow \vec{C}_1 = -\hat{\jmath} + 2\hat{k}$$

$$\therefore \vec{v}(t) = \frac{t^2}{2}\hat{\imath} + (e^t - 1)\hat{\jmath} + (2 - e^{-t})\hat{k}$$

$$\therefore \vec{r}(t) = \frac{t^3}{6}\hat{\imath} + (e^t - t)\hat{\jmath} + (2t + e^{-t})\hat{k} + \vec{C}_2$$

$$\vec{r}(0) = \hat{\jmath} + \hat{k} \Rightarrow \hat{\jmath} + \hat{k} = \hat{\jmath} + \hat{k} + \vec{C}_2$$

$$\Rightarrow \vec{C}_2 = \vec{0}\,.$$

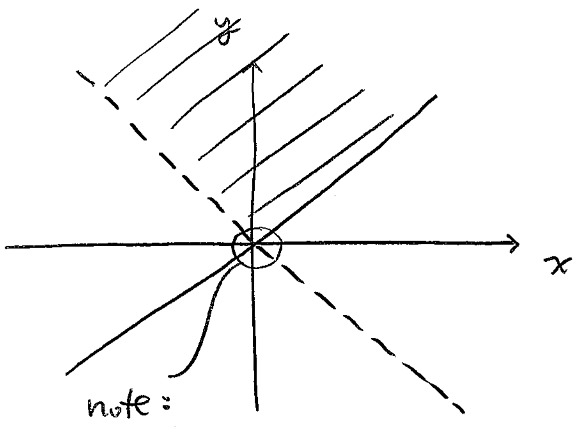
$$\boxed{\therefore \vec{r}(t) = \frac{t^3}{6}\hat{\imath} + (e^t - t)\hat{\jmath} + (2t + e^{-t})\hat{k}}$$

Question 2. [10]:

(a)[5] Determine and sketch the domain of $f(x,y) = \sqrt{y-x}\ln(y+x)$

$\sqrt{y-x}$ factor $\Rightarrow$ $y - x \geq 0 \Rightarrow y \geq x$

$\ln(y+x)$ factor $\Rightarrow$ $y + x > 0 \Rightarrow y > -x$



$\therefore$ domain is

$\{(x,y) \mid y \geq x \ \& \ y > -x\}$

(0,0) not included in domain

(b)[5] Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{6x^3y}{2x^4+y^4}$ does not exist.

- If $(x,y) \to (0,0)$ along positive x-axis, $y = 0$,

so $\dfrac{6x^3y}{2x^4+y^4} = \dfrac{0}{2x^4} = 0$ as $x \to 0$.

- If $(x,y) \to (0,0)$ along line $y = x$,

$\dfrac{6x^3y}{2x^4+y^4} = \dfrac{6x^4}{2x^4+x^4} = 2$ as $x \to 0$

Since different approach paths of (x,y) to $(0,0)$ produce different limiting values of $\dfrac{6x^3y}{2x^4+y^4}$) $\lim\limits_{(x,y)\to(0,0)} \dfrac{6x^3y}{2x^4+y^4}$ does not exist.

Question 3. [10]:

(a)[3] Let $f(x, y) = y \ln(x + 2y)$. Compute $f_x(1, 0) - f_y(1, 1)$.

$$f_x(x,y) = \frac{y}{x+2y} \; ; \; f_x(1,0) = \frac{0}{1+0} = 0$$

$$f_y(x,y) = \ln(x+2y) + \frac{2y}{x+2y} \; ; \; f_y(1,1) = \ln(1+2) + \frac{2(1)}{1+2}$$

$$= \ln(3) + \frac{2}{3}$$

$$\therefore \; f_x(1,0) - f_y(1,1) = 0 - \ln(3) - \frac{2}{3}$$

$$= \boxed{-\left(\ln(3) + \frac{2}{3}\right)}$$

(b)[3] Let $v = r\cos(s + 2t)$. Determine and simplify $\dfrac{\partial^2 v}{\partial t^2} + 4v$.

$$v_t = -2r\sin(s+2t)$$

$$v_{tt} = -4r\cos(s+2t).$$

$$v_{tt} + 4v = -4r\cos(s+2t) + 4r\cos(s+2t)$$

$$= \boxed{0}.$$

(c)[4] Use implicit differentiation to determine $\dfrac{\partial z}{\partial y}$ if

$$\sin(xyz) = x + 2y + 3z$$

$$\frac{\partial}{\partial y}\left[\sin(xyz)\right] = \frac{\partial}{\partial y}\left[x + 2y + 3z\right]$$

$$\cos(xyz)\, x \left[z + y\frac{\partial z}{\partial y}\right] = 2 + 3\frac{\partial z}{\partial y}$$

$$xz\cos(xyz) + xy\cos(xyz)\frac{\partial z}{\partial y} = 2 + 3\frac{\partial z}{\partial y}$$

$$\boxed{\therefore \; \frac{\partial z}{\partial y} = \frac{2 - xz\cos(xyz)}{xy\cos(xyz) - 3}}$$

Question 4. [10]:

(a)[5] Determine an equation of the tangent plane to the surface $z = e^x \cos(y)$ at the point where $x = 0$ and $y = 0$.

At $x=0, y=0$, $z = e^0 \cos(0) = 1$

$$\left.\frac{\partial z}{\partial x}\right|_{(0,0)} = \left. e^x \cos(y) \right|_{(0,0)} = 1$$

$$\left.\frac{\partial z}{\partial y}\right|_{(0,0)} = \left. -e^x \sin(y) \right|_{(0,0)} = 0$$

$$\therefore \quad z - z_0 = \frac{\partial z}{\partial x} \cdot (x - x_0) + \frac{\partial z}{\partial y}(y - y_0)$$

$$z - 1 = x$$

or $\boxed{z = x + 1}$

(b)[5] A box has height 80 cm , width 40 cm and length 50 cm. Use differentials (or a linear approximation) to determine the change in volume of the box if the height and width are both increased by 2 cm while the length is decreased by 1 cm .

$V = lwh$, $dl = -1$, $dw = 2$, $dh = 2$

$$dV = V_l\, dl + V_w\, dw + V_h\, dh$$

$$= wh\, dl + lh\, dw + lw\, dh$$

At $(l, w, h) = (50, 40, 80)$ and $(dl, dw, dh) = (-1, 2, 2)$:

$$dV = (40)(80)(-1) + (50)(80)(2) + (50)(40)(2)$$

$$= 8800 \text{ cm}^3.$$

$\therefore$ Volume <u>increases</u> by approx. 8800 cm^3.

Question 5 [10]:

(a)[5] Use the chain rule to determine $\frac{\partial v}{\partial t}$ at $(s,t) = (0,1)$ if

$$v = x^2 \sin(y) + ye^{xy}, \qquad x = s + 2t, \qquad y = st$$

$$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial t}$$

$$= \left(2x\sin(y) + y^2 e^{xy}\right)(2) + \left(x^2\cos(y) + e^{xy} + xye^{xy}\right)(s)$$

At $(s,t) = (0,1)$ we have $x = 0+2(1) = 2$, $y = (0)(1) = 0$,

$$\text{so } \frac{\partial v}{\partial t} = \Big[\underbrace{\cancel{(2)(2)\sin(0)}}_{0} + \underbrace{\cancel{0^2 e^{(2)(0)}}}_{0}\Big](2) + \Big[\underbrace{\cancel{2^2\cos(0) + e^{(2)(0)} + (2)(0)e^{(2)(0)}}}_{0}\Big](0)$$

$$= \boxed{0}$$

(b)[5] Is it possible for a function $f(x,y)$ to have first partial derivatives

$$f_x(x,y) = x + 4y \quad \text{and} \quad f_y(x,y) = 3x - y\ ?$$

Explain.

No, since otherwise

$$\left.\begin{aligned} &f_x(x,y) = x+4y \Rightarrow f_{xy}(x,y) = 4, \\ \text{yet } &f_y(x,y) = 3x-y \Rightarrow f_{yx}(x,y) = 3, \end{aligned}\right\} *$$

But by Clairaut's thm, $f_{xy} = f_{yx}$ which contradicts $*$.