Question 1. [10]:
For this question use

$$\mathbf{a} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}, \quad \mathbf{c} = \mathbf{j} - 5\mathbf{k}$$

(a)[3] Determine $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

$$\vec{b} \times \vec{c} = \langle 3,-2,1\rangle \times \langle 0,1,-5\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} = 9\hat{i} + 15\hat{j} + 3\hat{k}$$

$$\therefore \vec{a} \cdot (\vec{b} \times \vec{c}) = \langle 1,1,-2\rangle \cdot \langle 9,15,3\rangle$$
$$= 9 + 15 - 6$$
$$= 18$$

(b)[4] Determine $\text{proj}_{\mathbf{a}}\mathbf{b}$

$$\text{proj}_{\vec{a}} \vec{b} = \left(\vec{b} \cdot \frac{\vec{a}}{|\vec{a}|}\right) \frac{\vec{a}}{|\vec{a}|}$$

$$= \left(\langle 3,-2,1\rangle \cdot \frac{\langle 1,1,-2\rangle}{\sqrt{1^2+1^2+(-2)^2}}\right) \frac{\langle 1,1,-2\rangle}{\sqrt{1^2+1^2+(-2)^2}}$$

$$= \left(\frac{-1}{\sqrt{6}}\right) \frac{\langle 1,1,-2\rangle}{\sqrt{6}}$$

$$= \left\langle -\frac{1}{6}, \frac{-1}{6}, \frac{1}{3} \right\rangle$$

(c)[3] The angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$$\Rightarrow \cos\theta = \frac{\langle 1,1,-2\rangle \cdot \langle 3,-2,1\rangle}{\sqrt{1^2+1^2+(-2)^2}\sqrt{3^2+(-2)^2+1^2}}$$

$$= \frac{-1}{\sqrt{6}\sqrt{14}} = \frac{-1}{2\sqrt{21}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-1}{2\sqrt{21}}\right) \approx 96.3^\circ \quad \underline{\text{or}} \quad 1.68 \text{ rad.}$$

Question 2. [10]:

(a)[5] Find parametric equations for the line through the point $(1, 0, -1)$ and which is parallel to the line given by

$$\frac{x-4}{3} = \frac{y}{2} = z + 2$$

using the given line; $x=4 \Rightarrow y=0 \Rightarrow z=-2$

so $P_1(4,0,-2)$ is on the line

$x=7 \Rightarrow y=2 \Rightarrow z=-1$

so $P_2(7,2,-1)$ is on the line

$\therefore$ vector $\vec{v} = \overrightarrow{P_1P_2} = \langle 7-4, 2-0, -1-(-2)\rangle = \langle 3,2,1\rangle$

is parallel to given line.

$\therefore$ Line of interest has vector equation

$\vec{r} = \langle 1,0,-1\rangle + t\langle 3,2,1\rangle$

So parametric equations are

$x = 1+3t$, $y = 2t$, $z = -1+t$.

(b)[5] Find symmetric equations for the line through $Q(-2, 0, 8)$ and $R(1, 5, 0)$.

Direction vector is $\langle -2-1, 0-5, 8-0\rangle = \langle -3,-5,8\rangle$.

$\therefore$ vector equation is $\vec{r} = \langle -2,0,8\rangle + t\langle -3,-5,8\rangle$

$\therefore$ parametric equations are

$x = -2-3t \Rightarrow t = \frac{x+2}{-3}$

$y = -5t \Rightarrow t = \frac{y}{-5}$

$z = 8+8t \Rightarrow t = \frac{z-8}{8}$

$\therefore$ symmetric equations are

$$-\frac{x+2}{3} = -\frac{y}{5} = \frac{z-8}{8}.$$

Question 3. [10]:
For this question use the points $P(4,3,6)$, $Q(-2,0,8)$ and $R(1,5,0)$

(a)[5] Determine the area of the triangle with vertices P, Q and R.

Area $A = \frac{1}{2}\left|\vec{PQ} \times \vec{PR}\right|$

$$= \frac{1}{2}\left|\langle -2-4, -3, 8-6\rangle \times \langle 1-4, 5-3, 0-6\rangle\right|$$

$$= \frac{1}{2}\left|\langle -6,-3,2\rangle \times \langle -3,2,-6\rangle\right|$$

$$= \frac{1}{2}\left|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6 & -3 & 2 \\ -3 & 2 & -6 \end{vmatrix}\right|$$

$$= \frac{1}{2}\left|14\hat{i} - 42\hat{j} - 21\hat{k}\right|$$

$$= \frac{1}{2}\sqrt{14^2 + 42^2 + 21^2} = \boxed{\frac{49}{2}}$$

(b)[5] Show that the triangle with vertices P, Q and R is a right triangle.

$$\vec{PQ} \cdot \vec{PR} = \langle -6,-3,2\rangle \cdot \langle -3,2,-6\rangle$$

$$= 18 + (-6) + (-12)$$

$$= 0$$

$\therefore$ sides PQ and PR are $\perp$,

so PQR forms a right triangle.

Question 4. [10]:

(a)[5] Find an equation of the plane through $(2, 1, 0)$ and parallel to $x + 4y - 3z = 1$.

Normal to plane is $\vec{n} = \langle 1, 4, -3 \rangle$.

$\therefore$ Equation is $\left(\langle x, y, z\rangle - \langle 2, 1, 0\rangle\right) \cdot \vec{n} = 0$

$$\Rightarrow \quad (x-2) + 4(y-1) - 3z = 0$$

(b)[5] Find an equation of the plane through $(1, 2, -2)$ that contains the line

$$x = 2t, \quad y = 3 - t, \quad z = 1 + 3t$$

Use line to find two other points on the plane :

$t = 0 \Rightarrow x = 0,\ y = 3,\ z = 1$

$t = 1 \Rightarrow x = 2,\ y = 2,\ z = 4$

$\therefore$ $P(1, 2, -2)$, $Q(0, 3, 1)$, $R(2, 2, 4)$ are points on plane.

$\therefore$ $\vec{u} = \vec{PQ} = \langle -1, 1, 3\rangle$

and $\vec{v} = \vec{PR} = \langle 1, 0, 6\rangle$ are $\parallel$ to plane.

$\therefore$ $\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 3 \\ 1 & 0 & 6 \end{vmatrix} = \langle 6, 9, -1\rangle$ is normal to plane

$\therefore$ Plane is $\langle x-1, y-2, z+2\rangle \cdot \langle 6, 9, -1\rangle = 0$

$$\Rightarrow \quad 6(x-1) + 9(y-2) - (z+2) = 0.$$

Question 5 [10]:

(a)[5] Determine the distance between the planes $3x + y - 4z = 2$ and $3x + y - 4z = 24$.

a unit normal to both planes is

$$\vec{n} = \frac{\langle 3,1,-4\rangle}{\sqrt{3^2+1^2+(-4)^2}} = \frac{\langle 3,1,-4\rangle}{\sqrt{26}}.$$

A Point on first plane is $P_1(0,2,0)$,

A point on second plane is $P_2(0,24,0)$.

$\therefore$ distance between planes is

$$\left|\operatorname{Comp}_{\vec{n}} \overrightarrow{P_1P_2}\right| = \left|\overrightarrow{P_1P_2} \cdot \vec{n}\right|$$

$$= \langle 0,22,0\rangle \cdot \frac{\langle 3,1,-4\rangle}{\sqrt{26}}$$

$$= \frac{22}{\sqrt{26}}.$$

(b)[5] Find an equation for the set of all points equidistant from the points $P(7, -8, -9)$ and $Q(-4, 2, -10)$.

Let (x,y,z) be such a point. Then

$$(x-7)^2 + (y+8)^2 + (z+9)^2 = (x+4)^2 + (y-2)^2 + (z+10)^2$$

$$\cancel{x^2} - 14x + 49 + \cancel{y^2} + 16y + 64 + \cancel{z^2} + 18z + 81 = \cancel{x^2} + 8x + 16 + \cancel{y^2} - 4y + 4 + \cancel{z^2} + 20z + 100$$

$$\therefore \quad 22x - 20y + 2z = 74$$

$$11x - 10y + z = 37,$$