Question 1:

(a)[5] Let $y = x^2 \ln(x)$. Determine all values of x at which y"=0.

$$y' = 2x\ln(x) + x^2 \cdot \frac{1}{x} = 2x\ln(x) + x$$

$$y'' = 2\ln(x) + 2x \cdot \frac{1}{x} + 1 = 2\ln(x) + 3$$

$$y'' = 0 \Rightarrow 2\ln(x) + 3 = 0$$

$$\ln(x) = -\frac{3}{2}$$

$$\boxed{x = e^{-3/2}}$$

(b)[5] Given the demand equation $p = 250e^{(-q/50)}$, determine the point elasticity of demand when $q = 50$ and state whether the demand is elastic, inelastic, or unit elastic.

$$\frac{dp}{dq} = 250\, e^{-\frac{q}{50}} \cdot \left(-\frac{1}{50}\right) = -5e^{-\frac{q}{50}}$$

$$\eta = \frac{(p/q)}{\left(dp/dq\right)} = \frac{\left[\frac{250e^{-q/50}}{q}\right]}{\left[-5e^{-q/50}\right]} = \left(\frac{250}{q}\right)\left(\frac{-1}{5}\right) = \frac{-50}{q}.$$

$$\therefore \eta\Big|_{q=50} = \frac{-50}{50} = \boxed{-1}$$

$|\eta| = 1$, so demand is unit elastic.

Question 2:

(a)[5] Determine an equation of the tangent line to the curve $x\sqrt{y+1} = y\sqrt{x+1}$ at the point $(3,3)$.

$$\frac{d}{dx}\left[x(y+1)^{1/2}\right] = \frac{d}{dx}\left[y(x+1)^{1/2}\right]$$

$$(y+1)^{1/2} + x\frac{1}{2}(y+1)^{-\frac{1}{2}}\frac{dy}{dx} = \frac{dy}{dx}(x+1)^{1/2} + y\frac{1}{2}(x+1)^{-\frac{1}{2}}$$

at $(x,y) = (3,3)$:

$$(3+1)^{1/2} + 3\cdot\frac{1}{2}\cdot(3+1)^{-\frac{1}{2}}y' = y'(3+1)^{1/2} + 3\cdot\frac{1}{2}\cdot(3+1)^{-\frac{1}{2}}$$

$$2 + \frac{3}{4}y' = 2y' + \frac{3}{4}$$

$$\Rightarrow y'\left[2-\frac{3}{4}\right] = \left[2-\frac{3}{4}\right] \Rightarrow y' = 1$$

$\therefore$ equation is $\quad y - 3 = 1\cdot(x-3) \Rightarrow \boxed{y = x}$

(b)[5] Determine y' when $x = 1$ if $y = \left(\dfrac{3}{x^2}\right)^x$.
(Logarithmic differentiation may help here.)

$$\ln(y) = \ln\left[\left(\frac{3}{x^2}\right)^x\right]$$

$$\ln(y) = x\left[\ln(3) - \ln(x^2)\right]$$

$$\ln(y) = x\cdot\ln(3) - 2x\ln(x)$$

$$\Rightarrow \frac{1}{y}y' = \ln(3) - 2\ln(x) - \cancel{2x}\frac{1}{\cancel{x}}$$

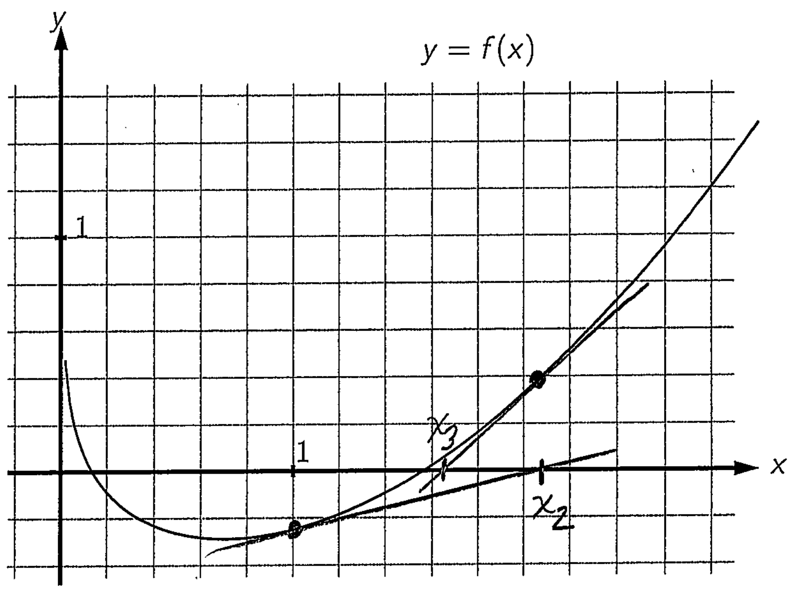
$$\Rightarrow y' = \left(\frac{3}{x^2}\right)^x\left[\ln(3) - 2 - 2\ln(x)\right]$$

at $x = 1$, $\quad y' = \left(\frac{3}{1^2}\right)^1\left[\ln(3) - 2 - 2\cancel{\ln(1)}^{0}\right]$

$$\boxed{y' = 3\left[\ln(3) - 2\right]}$$

Question 3:

(a)[5] We wish to find a solution to the equation $f(x) = 0$ using Newton's method, where the graph of $y = f(x)$ is shown below. Beginning with the starting value $x_1 = 1$, use a straight-edge (your student ID will do) to draw tangent lines on the graph to locate x_2 and x_3, the next two approximations given by Newton's method. Clearly indicate and label the points corresponding to x_2 and x_3. Your answer to this question consists of the tangent lines you draw and the points you indicate.



(b)[5] Suppose you wish to solve the equation $x^3 = 4$ using Newton's method. Beginning with a starting value of $x_1 = 1$, determine x_2 and x_3, the next two approximations to the solution of the equation.

$f(x) = x^3 - 4$

$f'(x) = 3x^2$

$\therefore x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^3 - 4}{3x_n^2}$.

n	x_n	x_{n+1}
1	1	$1 - \dfrac{1^3 - 4}{3 \cdot 1^2} = 2$
2	2	$2 - \dfrac{2^3 - 4}{3 \cdot 2^2} = \dfrac{5}{3}$

$$\boxed{\begin{aligned} \therefore\ x_2 &= 2 \\ x_3 &= \tfrac{5}{3} \end{aligned}}$$

Question 4:

(a)[5] Determine $f''(0)$ if $f(\theta) = e^{2\theta}\cos 3\theta$.

$$f'(\theta) = 2e^{2\theta}\cos(3\theta) + e^{2\theta}(-\sin(3\theta)\cdot 3)$$

$$= 2e^{2\theta}\cos(3\theta) - 3e^{2\theta}\sin(3\theta)$$

$$f''(\theta) = 4e^{2\theta}\cos(3\theta) + 2e^{2\theta}\cdot(-\sin(3\theta)\cdot 3)$$

$$- 6e^{2\theta}\sin(3\theta) - 3e^{2\theta}\cdot\cos(3\theta)\cdot 3$$

$$f''(0) = 4e^0\overbrace{\cos(0)}^{1} + 2e^0[\overbrace{-\sin(0)}^{0}\cdot 3] - 6e^0\overbrace{\sin(0)}^{0} - 3e^0\cdot\overbrace{\cos(0)}^{1}\cdot 3$$

$$= \boxed{-5}$$

(b)[5] Let $\bar{c} = \dfrac{20}{\ln q}$ be the average cost of producing q units of product, where $q > 1$. The total cost has a single relative minimum. Determine $\bar{c}$ at the value of q which minimizes total cost.

$$c = \bar{c}q = \frac{20q}{\ln(q)}.$$

$$\frac{dc}{dq} = \frac{\ln(q)\cdot 20 - 20q\cdot\frac{1}{q}}{[\ln(q)]^2} = \frac{20[\ln(q) - 1]}{[\ln(q)]^2}$$

$$\frac{dc}{dq} = 0 \Rightarrow \ln(q) - 1 = 0 \Rightarrow \ln(q) = 1 \Rightarrow q = e$$

$$\bar{C}\Big|_{q=e} = \frac{20}{\ln(q)}\Big|_{q=e} = \frac{20}{\cancel{\ln(e)}\,1} = \boxed{20}$$

Question 5: For this question use the function $f(x) = x^2e^x$

(a)[7] Determine the intervals of increase and decrease of $f(x)$. State a clear conclusion.

$$f'(x) = 2xe^x + x^2e^x = xe^x[2+x].$$

- $\underline{f'(x)=0}$? $x(2+x)e^x = 0 \Rightarrow x=0,\ x=-2.$
- $\underline{f'(x) \text{ not exist}}$? no such x.

always ⊕

	-3	-2	-1	0	1
$f'(x) = e^x x(x+2)$:	$+$	0	$-$	0	$+$
$f(x) = x^2e^x$:	$\nearrow$	$4e^{-2}$	$\searrow$	0	$\nearrow$

$\therefore$ f is increasing on $(-\infty,-2) \cup (0,\infty)$.
f is decreasing on $(-2,0)$.

(b)[3] State the relative extrema of $f(x)$.

f has a rel. max. of $4e^{-2}$ at $x=-2$;
f has a rel. min. of 0 at $x=0$.