Question 1:

(a)[5 points] Use T_6, the trapezoid rule on six subintervals, to approximate $\int_0^{3\pi} \underbrace{x^2 \cos x}_{f(x)}\, dx$.

$\Delta x = \frac{3\pi}{6} = \frac{\pi}{2}$

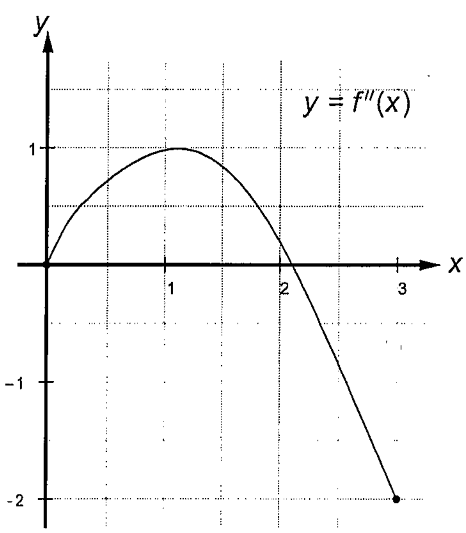
$\therefore T_6 = \frac{\Delta x}{2}\left[\cancelto{0}{f(0)} + 2\cancelto{0}{f(\tfrac{\pi}{2})} + 2f(\pi) + 2\cancelto{0}{f(\tfrac{3\pi}{2})} + 2f(2\pi) + 2\cancelto{0}{f(\tfrac{5\pi}{2})} + f(3\pi)\right]$

$= \frac{\pi}{4}\left[2\pi^2 \cancelto{-1}{\cos(\pi)} + 2(2\pi)^2 \cancelto{1}{\cos(2\pi)} + (3\pi)^2 \cancelto{-1}{\cos(3\pi)}\right]$

$= \frac{\pi}{4}\left[\pi^2(-2+8-9)\right]$

$= \boxed{-\frac{3\pi^3}{4}}$

(b)[5 points] The graph of $y = f''(x)$ is shown below. If the trapezoid rule is used to approximate $\int_0^3 f(x)\,dx$, how many subintervals are required to ensure that the error in our approximation is less than 1/8. Recall, the error in using the trapezoid rule to approximate $\int_a^b f(x)\,dx$ is at most $\frac{K(b-a)^3}{12n^2}$, where $|f''(x)| \leqslant K$ on $[a, b]$.



$-2 \leqslant f''(x) \leqslant 1$, so $|f''(x)| \leqslant 2 = K$.

We need

$\frac{2(3-0)^3}{12 \cdot n^2} \leqslant \frac{1}{8}$

$\Rightarrow \frac{2 \cdot 3^3 \cdot 8}{12} \leqslant n^2$

$\Rightarrow 2^2 \cdot 3^2 \leqslant n^2$

$\Rightarrow \boxed{n \geqslant 6}$

Question 2:

(a)[5 points] Evaluate the improper integral or show that is it divergent. Make proper use of any required limits.

$$I = \int_3^5 \frac{x}{\sqrt{x^2-9}}\,dx$$

$$= \lim_{t\to 3^+} \int_t^5 \frac{x}{\sqrt{x^2-9}}\,dx \quad \left.\begin{array}{l} \text{let } u = x^2-9 \\ du = 2x\,dx \end{array}\right\}$$

$$= \lim_{t\to 3^+} \cancel{\left(\tfrac{1}{2}\right)} \frac{\sqrt{x^2-9}}{\cancel{\left(\tfrac{1}{2}\right)}} \Bigg|_t^5$$

$$= \lim_{t\to 3^+} \sqrt{16} - \sqrt{t^2-9}$$

$$= 4 \qquad \therefore \text{ integral converges to } 4.$$

(b)[5 points] Evaluate the improper integral or show that it is divergent.

$$I = \int_{-\infty}^{\infty} \frac{x^5}{1+3x^6}\,dx = \int_{-\infty}^{0} \frac{x^5}{1+3x^6}\,dx + \int_0^{\infty} \frac{x^5}{1+3x^6}\,dx$$

For $I_1 = \int_0^\infty \frac{x^5}{1+3x^6}\,dx$, let $u = 1+3x^6$, $du = 18x^5\,dx$

$$\therefore I_1 = \lim_{t\to\infty} \int_0^t \frac{x^5}{1+3x^6}\,dx = \lim_{t\to\infty} \frac{1}{18} \ln|1+3x^6| \Big|_0^t$$

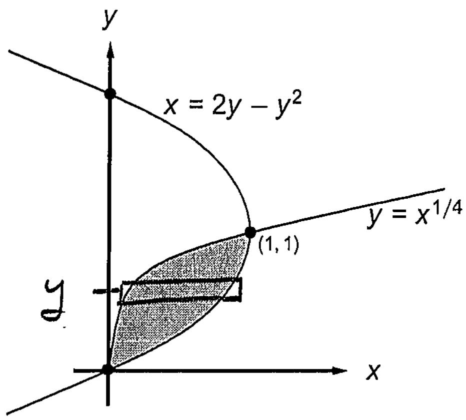
$$= \lim_{t\to\infty} \frac{1}{18}\left[\ln|1+3t^6| - \cancel{\ln|1|}^{\,0}\right]$$

$$= \infty.$$

Since $\int_0^\infty \frac{x^5}{1+3x^6}\,dx$ diverges, so does $\int_{-\infty}^{\infty} \frac{x^5}{1+3x^6}\,dx$.

Question 3:

(a)[5 points] Find the area of the shaded region below:



$\} \Rightarrow x = y^4$

$$A = \int_0^1 (2y - y^2) - y^4 \, dy$$

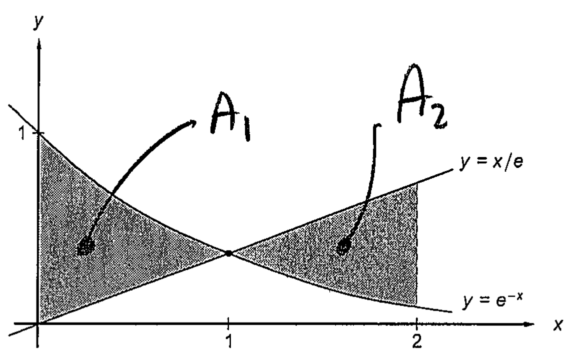
$$= \left[\frac{2y^2}{2} - \frac{y^3}{3} - \frac{y^5}{5}\right]_0^1$$

$$= 1 - \frac{1}{3} - \frac{1}{5}$$

$$= \frac{15 - 5 - 3}{15}$$

$$= \boxed{\frac{7}{15}}$$

(b)[5 points] Find the area of the shaded region below:



$$A_1 = \int_0^1 e^{-x} - \frac{x}{e} \, dx$$

$$= \left[-e^{-x} - \frac{x^2}{2e}\right]_0^1$$

$$= -e^{-1} - \frac{1}{2e} + 1$$

$$= -\frac{3}{2e} + 1$$

$$A_2 = \int_1^2 \frac{x}{e} - e^{-x} \, dx$$

$$= \left[\frac{x^2}{2e} + e^{-x}\right]_1^2$$

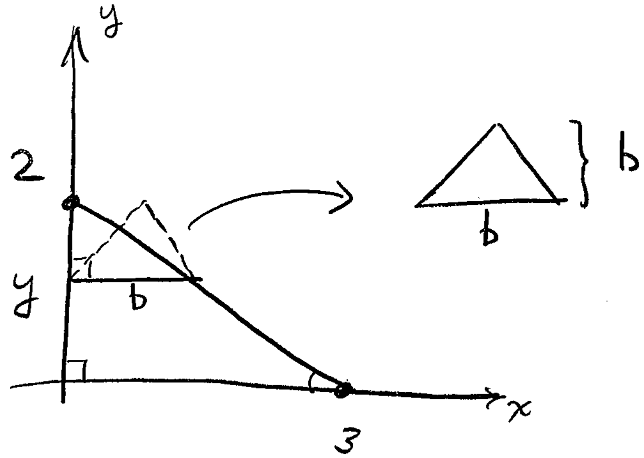
$$= \frac{2}{e} + \frac{1}{e^2} - \frac{1}{2e} - \frac{1}{e} = \frac{4e + 2 - e - 2e}{2e^2}$$

$$\therefore A_1 + A_2 = \frac{1}{2} - \frac{1}{e} + \frac{2}{e} + \frac{1}{e^2} - \frac{1}{2e} - \frac{1}{e}$$

$$= \frac{2e^2 - 3e + 4e + 2 - e - 2e}{2e^2}$$

$$= \boxed{\frac{2e^2 - 2e + 2}{2e^2}}$$

Question 4 [10 points]: The base (flat bottom surface) of the solid S is the triangular region with vertices at $(0,0)$, $(3,0)$ and $(0,2)$. Cross-sections perpendicular to the y-axis are isosceles triangles of equal base and height. Find the volume of S.



$\therefore A = \frac{1}{2} \cdot b \cdot b = \frac{1}{2} b^2.$

By similar triangles,

$$\frac{2-y}{2} = \frac{b}{3}$$

$\therefore \quad b = \frac{3}{2}(2-y).$

$$\therefore A(y) = \frac{1}{2}\left[\frac{3}{2}(2-y)\right]^2 = \frac{9}{8}(2-y)^2.$$

$$\therefore V = \int_0^2 \frac{9}{8}(2-y)^2\,dy$$

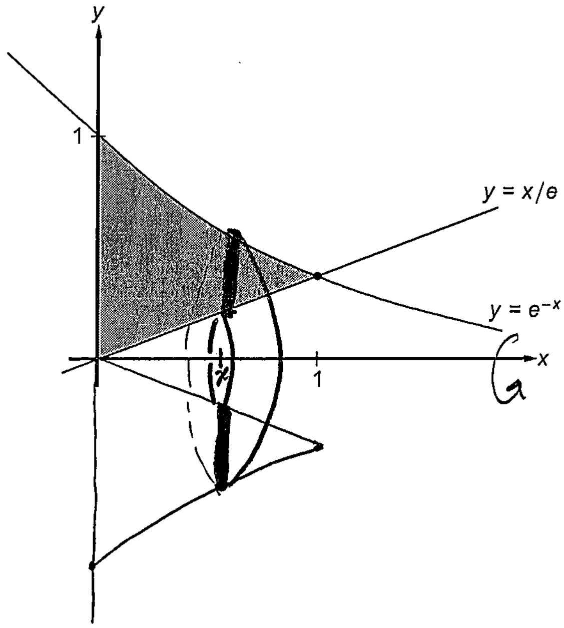
$$= \frac{9}{8}\left(\frac{-1}{3}\right)(2-y)^3 \Big|_0^2$$

$$= \frac{-3}{8}\left(0 - 2^3\right)$$

$$= \boxed{3}$$

Question 5:

(a)[5 points] Set up but do not evaluate the integral representing the volume of the solid obtained by rotating the shaded region about the x-axis. (Use the washer or, equivalently, the disk method.)

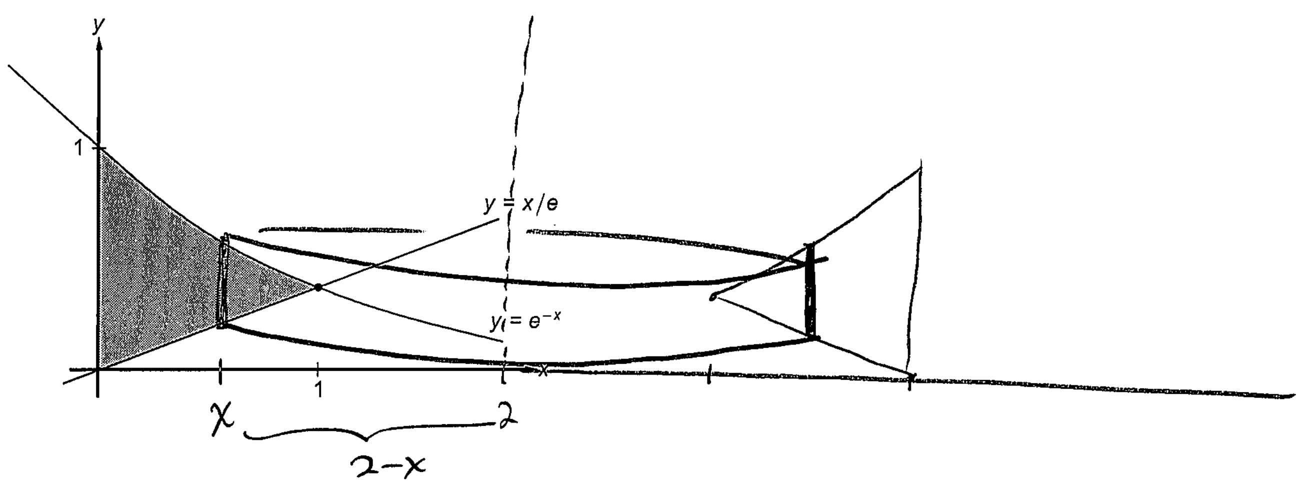


$$A(x) = \pi (e^{-x})^2 - \pi \left(\frac{x}{e}\right)^2$$

$$= \pi \left[e^{-2x} - \frac{x^2}{e^2} \right]$$

$$\therefore V = \int_0^1 \pi \left[e^{-2x} - \frac{x^2}{e^2} \right] dx$$

(a)[5 points] Set up but do not evaluate the integral representing the volume of the solid obtained by rotating the shaded region about the line $x = 2$. (Use cylindrical shells.)



$$V = \int_0^1 2\pi (2-x)\left(e^{-x} - \frac{x}{e}\right) dx.$$