Question 1 [10 points]: Evaluate

$$I = \int \frac{8x}{(x-2)(x+2)^2}\,dx$$

$$\frac{8x}{(x-2)(x+2)^2} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$= \frac{A(x+2)^2 + B(x-2)(x+2) + C(x-2)}{(x-2)(x+2)^2}$$

$$= \frac{(A+B)x^2 + (4A+C)x + 4A-4B-2C}{(x-2)(x+2)^2}$$

$\therefore\ A+B=0 \Rightarrow B=-A$

$4A+C=8 \Rightarrow C = 8-4A$

$4A-4B-2C=0 \Rightarrow 4A-4(-A)-2(8-4A)=0$

$16A-16=0$

$A = \frac{16}{16} = 1$

$\therefore\ B=-A=-1$

$\therefore\ C = 8-4A = 8-4=4$

$$\therefore\ I = \int \frac{1}{x-2} - \frac{1}{x+2} + \frac{4}{(x+2)^2}\,dx$$

$$= \boxed{\ln|x-2| - \ln|x+2| - 4\frac{1}{(x+2)} + C}$$

Question 2:

(a)[6] Use T_4, the trapezoid rule on 4 subintervals, to approximate $\int_{-1}^{1} 2e^{\sin(\pi x)}\,dx$.

$$\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{1}{2}$$

$$f(x) = 2e^{\sin(\pi x)}$$

$$\int_{-1}^{1} 2e^{\sin(\pi x)}dx \approx T_4 = \frac{\Delta x}{2}\left[f(-1) + 2f(-\tfrac{1}{2}) + 2f(0) + 2f(\tfrac{1}{2}) + f(1)\right]$$

$$= \frac{1}{4}\left[2 + 4e^{-1} + 4 + 4e + 2\right]$$

$$= \boxed{2 + e^{-1} + e}$$

(b)[4] For the integrand $f(x) = 2e^{\sin(\pi x)}$ in part (a),

$$f''(x) = 2\pi^2 e^{\sin(\pi x)}\left[\cos^2(\pi x) - \sin(\pi x)\right] .$$

On the interval $[-1, 1]$ this second derivative has an absolute maximum at $x = 0$ and an absolute minimum at $x = 1/2$. Use this information to determine an error bound on your approximation in part (a).

Recall: the error in using the trapezoid rule to approximate $\int_a^b f(x)\,dx$ using n subintervals is at most $\frac{K(b-a)^3}{12n^2}$ where $|f''(x)| \le K$ on $[a, b]$.

$$|f''(0)| = \left|2\pi^2 e^{\overset{0}{\cancel{\sin(0)}}}\left[\overset{1}{\cancel{\cos^2(0)}} - \underset{0}{\cancel{\sin(0)}}\right]\right| = 2\pi^2$$

$$|f''(\tfrac{1}{2})| = \left|2\pi^2 e^{\overset{1}{\cancel{\sin(\frac{\pi}{2})}}}\left[\cos^2\underset{0}{\cancel{(\tfrac{\pi}{2})}} - \underset{1}{\cancel{\sin(\tfrac{\pi}{2})}}\right]\right| = 2e\pi^2$$

$\therefore\ K = 2e\pi^2$ and

$$E_{T_4} \le \frac{2e\pi^2(1-(-1))^3}{(12)(4^2)} = \boxed{\frac{e\pi^2}{12}}$$

Question 3:

(a)[5] Evaluate the improper integral. Clearly show all steps including any required limits, and state, based on your answer, whether the integral converges or diverges:

$$\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx$$

For $I = \int \frac{e^x}{e^{2x}+1}\,dx$, let $u = e^x$, $du = e^x\,dx$.

$\therefore\ I = \int \frac{1}{1+u^2}\,du = \arctan(u) + C = \arctan(e^x) + C,$

$$\therefore \int_0^\infty \frac{e^x}{e^{2x}+1}\,dx = \lim_{t\to\infty} \int_0^t \frac{e^x}{e^{2x}+1}\,dx$$

$$= \lim_{t\to\infty} \left[\arctan(e^x)\right]_0^t$$

$$= \lim_{t\to\infty} \left[\arctan(e^t) - \arctan(\underbrace{e^0}_{1})\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$\therefore$ integral converges to $\frac{\pi}{4}$.

(b)[5] Determine (with justification) whether $\int_1^\infty \frac{\cos^4 x + 4}{x^4}\,dx$ converges or diverges.

On $[1,\infty)$, $0 \leq \frac{\cos^4 x + 4}{x^4} \leq \frac{5}{x^4}$.

Since $\int_1^\infty \frac{5}{x^4}\,dx = 5\int_1^\infty \frac{1}{x^4}\,dx$ converges (p-integral, $p > 1$),

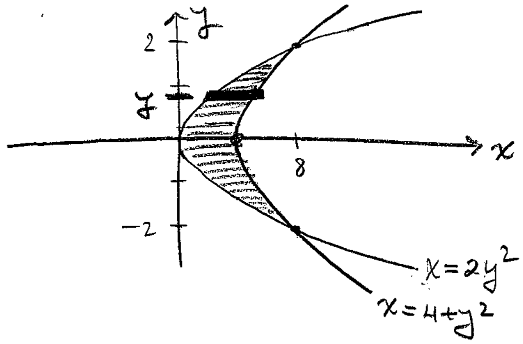
then by the comparison theorem,

$$\int_1^\infty \frac{\cos^4 x + 4}{x^4}\,dx$$

also converges.

Question 4:

(a)[5] Determine the area of the region bounded by the two parabolas $x = 2y^2$ and $x = 4 + y^2$:



$$2y^2 = 4+y^2$$
$$y^2 = 4$$
$$y = \pm 2$$

$$A = \int_{-2}^{2} (4+y^2) - (2y^2)\, dy$$

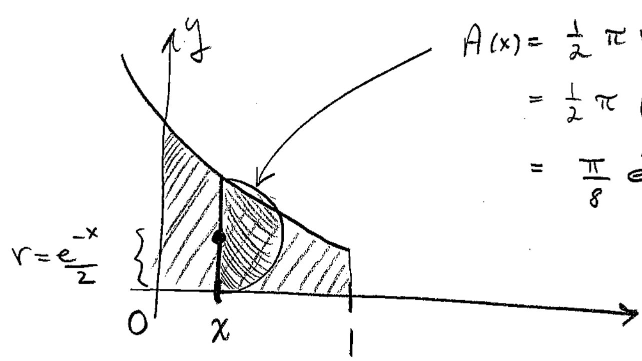
$$= 2\int_0^2 4 - y^2\, dy$$

$$= 2\left[4y - \frac{y^3}{3}\right]_0^2$$

$$= 2\left(8 - \frac{8}{3}\right)$$

$$= \boxed{\frac{32}{3}}$$

(b)[5] The base (flat bottom) of a solid is the region in the xy-plane bounded by the curves $y = e^{-x}$, $x = 0$ and $x = 1$. Cross-sections perpendicular to the x-axis are semicircles. Determine the volume of the solid.



$$A(x) = \frac{1}{2}\pi r^2$$
$$= \frac{1}{2}\pi\left(\frac{e^{-x}}{2}\right)^2$$
$$= \frac{\pi}{8} e^{-2x}$$

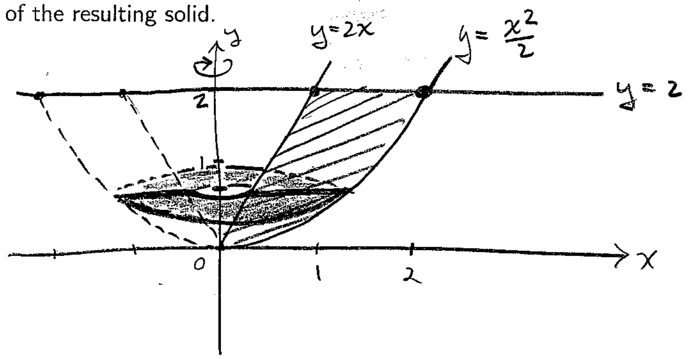
$$\therefore V = \int_0^1 A(x)\, dx$$

$$= \int_0^1 \frac{\pi}{8} e^{-2x}\, dx$$

$$= \frac{\pi}{8}\left[\frac{e^{-2x}}{-2}\right]_0^1 = \frac{\pi}{8}\left[\frac{e^{-2}}{-2} - \frac{1}{-2}\right] = \boxed{\frac{\pi}{16}\left(1 - e^{-2}\right)}$$

Question 5:

(a)[5] The region bounded by the curves $y = 2x$, $y = x^2/2$ and $y = 2$ is rotated about the y-axis. Determine the volume of the resulting solid.



By washer method:

$y = 2x \Rightarrow x = \frac{y}{2}$

$y = \frac{x^2}{2} \Rightarrow x = \sqrt{2}\sqrt{y}$

$A(y) = \pi(\sqrt{2}\sqrt{y})^2 - \pi\left(\frac{y}{2}\right)^2 = \pi\left[2y - \frac{y^2}{4}\right]$

$\therefore V = \int_{y=0}^{y=2} \pi\left[2y - \frac{y^2}{4}\right] dy$

$= \pi\left[\frac{\cancel{2}y^2}{\cancel{2}} - \frac{y^3}{12}\right]_0^2$

$= \pi\left[4 - \frac{8}{12}\right]$

$= \boxed{\frac{10\pi}{3}}$

(b)[5] Determine the length of the curve $y = 2\ln(\cos(x/2))$, $0 \le x \le \pi/2$.

$L = \int_0^{\frac{\pi}{2}} \sqrt{1 + (y')^2}\, dx$

$= \int_0^{\frac{\pi}{2}} \sqrt{1 + \left(\cancel{2}\frac{1}{\cos(\frac{x}{2})} \cdot \sin\left(\frac{x}{2}\right) \cdot \cancel{\frac{1}{2}}\right)^2}\, dx$

$= \int_0^{\frac{\pi}{2}} \sqrt{1 + \tan^2\left(\frac{x}{2}\right)}\, dx$

$= \int_0^{\frac{\pi}{2}} \sec\left(\frac{x}{2}\right) dx$

$= 2\left[\ln\left|\sec\left(\frac{x}{2}\right) + \tan\left(\frac{x}{2}\right)\right|\right]_0^{\frac{\pi}{2}}$

$= 2\ln\left|\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right| - 2\ln\left|\overset{1}{\cancel{\sec(0)}} + \overset{0}{\cancel{\tan(0)}}\right| \to 0$

$= \boxed{2\ln(\sqrt{2} + 1)}$