Question 1 [10 points]: Use the definition of the definite integral in the form

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

to evaluate

$$\int_0^2 (x^2 - x)\,dx$$

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

$$x_i = a + i\Delta x = 0 + i\left(\frac{2}{n}\right) = \frac{2i}{n}.$$

$$\therefore \int_0^2 (x^2 - x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{2i}{n}\right)^2 - \left(\frac{2i}{n}\right)\right]\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{4i^2}{n^2} - \frac{2i}{n}\right]\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{8i^2}{n^3} - \frac{4i}{n^2}\right)$$

$$= \lim_{n\to\infty} \left[\frac{8}{n^3}\sum_{i=1}^{n} i^2 - \frac{4}{n^2}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \left[\frac{8}{n^3} \frac{n(n+1)(2n+1)}{6} - \frac{4}{n^2} \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \left[\frac{4}{3} \cdot \cancel{\frac{n}{n}}^{1} \cdot \underbrace{\frac{n+1}{n}}_{\downarrow\, 1} \cdot \underbrace{\frac{2n+1}{n}}_{\downarrow\, 2} - 2 \cdot \cancel{\frac{n}{n}}^{1} \cdot \underbrace{\frac{n+1}{n}}_{\downarrow\, 1}\right]$$

$$= \frac{4}{3}\cdot 1 \cdot 2 - 2 \cdot 1$$

$$= \boxed{\frac{2}{3}}$$

Check: $\int_0^2 x^2 - x\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^2 = \frac{8}{3} - \frac{4}{2} = \frac{2}{3}$.

Question 2:

(a)[3] Determine the average value of $f(x) = \frac{\ln x}{x}$ over the interval $[1, e]$.

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{e-1}\int_1^e \frac{\ln x}{x}\,dx \quad \left.\right\} \begin{array}{l} \text{let } u = \ln x \\ du = \frac{1}{x}dx \end{array} \left\{ \begin{array}{l} x=1 \Rightarrow u = 0 \\ x = e \Rightarrow u = 1 \end{array} \right.$$

$$= \frac{1}{e-1}\int_0^1 u\,du$$

$$= \frac{1}{e-1}\left[\frac{u^2}{2}\right]_0^1 = \boxed{\frac{1}{2(e-1)}}.$$

(b)[3] Evaluate $\int_1^2 \frac{2}{3-5x}\,dx = I$

$$\begin{array}{l} \text{Let } u = 3-5x \\ du = -5\,dx \end{array} \left\{ \begin{array}{l} x=1 \Rightarrow u = -2 \\ x = 2 \Rightarrow u = -7 \end{array} \right.$$

$$\therefore I = \frac{-1}{5}\int_{-2}^{-7} \frac{2}{u}\,du$$

$$= -\frac{2}{5}\Big[\ln|u|\Big]_{-2}^{-7}$$

$$= \boxed{-\frac{2}{5}\left[\ln(7) - \ln(2)\right]} \quad \text{or} \quad \boxed{-\frac{2}{5}\ln\left(\frac{7}{2}\right)}$$

(c)[4] Evaluate $\int \sec^2 x \cos(\tan x)\,dx$.

$$\begin{array}{l} \text{Let } u = \tan x \\ du = \sec^2 x\,dx \end{array}$$

$$\therefore \int \sec^2 x \cos(\tan x)\,dx$$

$$= \int \cos(u)\,du$$

$$= \sin(u) + C$$

$$= \boxed{\sin(\tan x) + C}$$

Question 3 [10 points]: Evaluate

$$I = \int (x^2+1)\cos(2x)\,dx$$

$$u = x^2+1, \quad dv = \cos(2x)\,dx$$

$$du = 2x\,dx, \quad v = \frac{\sin(2x)}{2},$$

$$\therefore I = \int u\,dv$$

$$= uv - \int v\,du$$

$$= (x^2+1)\frac{\sin(2x)}{2} - \underbrace{\int \frac{\sin(2x)}{\cancel{2}} \cdot \cancel{2}x\,dx}$$

$$u = x \quad dv = \sin(2x)\,dx$$

$$du = dx \quad v = \frac{-\cos(2x)}{2}$$

$$= (x^2+1)\frac{\sin(2x)}{2} - \int u\,dv$$

$$= (x^2+1)\frac{\sin(2x)}{2} - \left[uv - \int v\,du\right]$$

$$= (x^2+1)\frac{\sin(2x)}{2} - \left[-\frac{x\cos(2x)}{2} - \int -\frac{\cos(2x)}{2}\,dx\right]$$

$$= (x^2+1)\frac{\sin(2x)}{2} + \frac{x\cos(2x)}{2} - \frac{1}{2}\int \cos(2x)\,dx$$

$$= \boxed{(x^2+1)\frac{\sin(2x)}{2} + \frac{x\cos(2x)}{2} - \frac{1}{4}\sin(2x) + C}$$

Question 4 [10 points]: Evaluate

$$\int_0^{\pi/3} \tan^5 x \sec^4 x \, dx$$

First determine $I = \int \tan^5 x \sec^4 x \, dx$

$$= \int \tan^5 x \sec^2 x \sec^2 x \, dx$$

$$= \int \tan^5 x \left(1+\tan^2 x\right) \sec^2 x \, dx$$

let $u = \tan x$

$du = \sec^2 x \, dx$

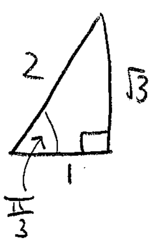
$$\therefore I = \int u^5 \left(1+u^2\right) du$$

$$= \int u^5 + u^7 \, du$$

$$= \frac{u^6}{6} + \frac{u^8}{8} + C$$

$$= \frac{\tan^6 x}{6} + \frac{\tan^8 x}{8} + C.$$

$$\therefore \int_0^{\frac{\pi}{3}} \tan^5 x \sec^4 x \, dx = \Big[I \Big]_0^{\frac{\pi}{3}}$$



$$= \frac{\tan^6\left(\frac{\pi}{3}\right)}{6} + \frac{\tan^8\left(\frac{\pi}{3}\right)}{8} - \cancel{\frac{\tan^6(0)}{6}} - \cancel{\frac{\tan^8(0)}{8}}$$

0 0

$$= \frac{\left(\sqrt{3}\right)^6}{6} + \frac{\left(\sqrt{3}\right)^8}{8}$$

$$= \frac{27}{6} + \frac{81}{8}$$

$$= \boxed{\frac{117}{8}}$$

Question 5: Determine

$$I = \int \frac{1}{x\sqrt{5-x^2}}\,dx$$

Let $x = \sqrt{5}\sin\theta$

$dx = \sqrt{5}\cos\theta\,d\theta$

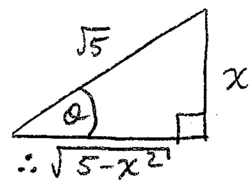
$$\therefore\quad I = \int \frac{1}{\sqrt{5}\sin\theta\sqrt{5-5\sin^2\theta}}\sqrt{5}\cos\theta\,d\theta$$

$$= \int \frac{1}{\sqrt{5}\sin\theta\,\cancel{\sqrt{5}}\,\cancel{\cos\theta}}\cdot\cancel{\sqrt{5}}\,\cancel{\cos\theta}\,d\theta$$

$$= \frac{1}{\sqrt{5}}\int \csc\theta\,d\theta$$

$$= \frac{1}{\sqrt{5}}\ln\left|\csc\theta - \cot\theta\right| + C.$$

$$x = \sqrt{5}\sin\theta \Rightarrow \sin\theta = \frac{x}{\sqrt{5}} \quad \therefore$$



$$\therefore\quad \boxed{I = \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}}{x} - \frac{\sqrt{5-x^2}}{x}\right| + C}$$