**Question 1:** Recall Newton's Law of Cooling and Heating: if $T(t)$ is the temperature of an object at time $t$ and $T_S$ is the temperature of the surroundings, then

$$\frac{dT}{dt} = k(T - T_S) \, ,$$

and

$$T(t) = T_S + (T_0 - T_S)e^{kt} \, .$$

Here $k$ is a negative constant and $T_0 = T(0)$, the initial temperature of the object.

**(a)[5]** A cup of coffee has an initial temperature of 95°C in a 20°C room. At the instant that the cup of coffee reaches 70°C it is cooling at a rate of 1°C per minute. Determine the value of the constant $k$ using this information.

$$\frac{dT}{dt} = k(T - T_s)$$

$$-1 = k(70 - 20)$$

$$\boxed{\therefore k = \frac{-1}{50}}$$

**(b)[5]** Using your result from part (a), determine the time at which the cup of coffee reaches the 70°C temperature.

Solve $T(t) = 70$ for $t$:

$$70 = 20 + (95 - 20)e^{-\frac{1}{50}t}$$

$$50 = 75 e^{-\frac{1}{50}t}$$

$$\frac{50}{75} = e^{-\frac{1}{50}t}$$

$$\ln\left(\frac{2}{3}\right) = -\frac{1}{50}t$$

$$\boxed{t = -50 \ln\left(\frac{2}{3}\right) \text{ minutes.}}$$

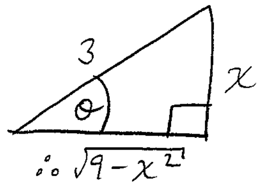
**Question 2:**

**(a)[3]** Write as a simplified expression which does not involve trigonometric or inverse trigonometric functions:

$$\cot(\arcsin(x/3))$$

Let $\theta = \arcsin\left(\frac{x}{3}\right)$

$\therefore \sin\theta = \frac{x}{3}$ $\therefore$




$\therefore \cot\left(\arcsin\left(\frac{x}{3}\right)\right)$

$= \cot\theta$

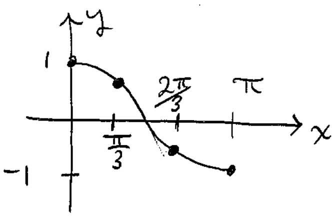
$= \boxed{\dfrac{\sqrt{9-x^2}}{x}}$

**(b)[3]** Determine the exact value of $\cos^{-1}(-1/2)$ .

$\cos^{-1}\left(\frac{-1}{2}\right) =$ angle $0 \le \theta \le \pi$ such that $\cos(\theta) = \frac{-1}{2}$

Using fact that $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and graph

$y = \cos x$ :




We have $\cos^{-1}\left(\frac{-1}{2}\right) = \boxed{\dfrac{2\pi}{3}}$

**(c)[4]** Let $f(x) = \arccos(\arcsin(x^2))$. Compute $f'(0)$.

$$f'(x) = \frac{-1}{\sqrt{1-[\arcsin(x^2)]^2}} \cdot \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$$

$$f'(0) = \frac{-1}{\sqrt{1-[\underbrace{\arcsin(0)}_{0}]^2}} \cdot \frac{1}{\sqrt{1-0^2}} \cdot 0$$

$= (-1)(1)(0)$

$= \boxed{0}$

**Question 3:**

**(a)[3]** Evaluate $\lim_{x\to\infty} e^{-2x}\sinh(2x)$.

$$\lim_{x\to\infty} e^{-2x}\sinh(2x)$$

$$= \lim_{x\to\infty} e^{-2x}\left(\frac{e^{2x}-e^{-2x}}{2}\right)$$

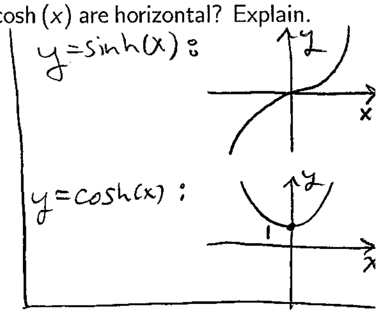
$$= \lim_{x\to\infty} \frac{1-e^{-4x}}{2}$$

$$= \boxed{\frac{1}{2}}$$

**(b)[3]** Are there any values of $x>0$ for which tangent lines to $y=\frac{\sinh^2(x)}{2}-\cosh(x)$ are horizontal? Explain.

$$y' = \frac{\cancel{2}\sinh(x)\cosh(x)}{\cancel{2}} - \sinh(x)$$

$$= \sinh(x)\left[\cosh(x)-1\right].$$




Tangent line horizontal $\Rightarrow y'=0$

$\Rightarrow \sinh(x)=0$ or $\cosh(x)=1$

But $\sinh(x)>0$ for $x>0$ and $\cosh(x)>1$ for $x>0$,

So there are no $x>0$ for which tangent lines are horizontal.

**(c)[4]** Solve for $x$: $\sinh(x)=1$.

$$\frac{e^x-e^{-x}}{2}=1$$

$$e^x-e^{-x}=2$$

$$e^{2x}-2e^x-1=0$$

$$(e^x)^2-2(e^x)-1=0$$

Quadratic in $e^x$:

$$e^x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-1)}}{2(1)}$$

$$=\frac{2\pm2\sqrt{2}}{2} = 1+\sqrt{2},\ 1-\sqrt{2}$$

since $1-\sqrt{2}<0$, we must have

$$e^x = 1+\sqrt{2},$$

$$\boxed{\therefore\ x=\ln\left(1+\sqrt{2}\right).}$$

**Question 4:**

**(a)[5]** Evaluate $\lim_{x\to 0} \frac{1-\cos(x)}{x^2+x}$ $\sim$ "$\frac{0}{0}$"

$$\overset{H}{=} \lim_{x\to 0} \frac{\sin(x)}{2x+1}$$

$$= \boxed{0}$$

**(b)[5]** Evaluate $\lim_{x\to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$ $\sim$ "$\infty - \infty$"

$$= \lim_{x\to 1^+} \frac{x\ln x - x + 1}{(x-1)\ln x} \sim \text{"}\frac{0}{0}\text{"}$$

$$\overset{H}{=} \lim_{x\to 1^+} \frac{\ln x + \cancel{x \cdot \frac{1}{x}} \cancel{-1}}{\ln x + (x-1)\cdot\frac{1}{x}}$$

$$= \lim_{x\to 1^+} \frac{\ln x}{\ln x + 1 - \frac{1}{x}} \sim \text{"}\frac{0}{0}\text{"}$$

$$\overset{H}{=} \lim_{x\to 1^+} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}}$$

$$= \boxed{\frac{1}{2}}$$

**Question 5:**

**(a)[5]** Evaluate $\lim_{x\to\infty} x\tan(1/x)$ $\sim$ "$\infty \cdot 0$"

$$\lim_{x\to\infty} x\tan\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}} \sim \text{"}\frac{0}{0}\text{"}$$

$$\stackrel{H}{=} \lim_{x\to\infty} \frac{\sec^2\left(\frac{1}{x}\right)\cdot\cancel{\left(\frac{-1}{x^2}\right)}}{\cancel{\left(\frac{-1}{x^2}\right)}}$$

$$= \sec^2(0)$$

$$= \frac{1}{\cos^2(0)}$$

$$= \boxed{1}$$

**(b)[5]** Evaluate $\lim_{x\to 0^+} x^{\sqrt{x}}$ $\sim$ "$0^0$"

$$x^{\sqrt{x}} = e^{\sqrt{x}\ln x} \therefore$$

now consider $\lim_{x\to 0^+} \sqrt{x}\ln x \sim$ "$0\cdot\infty$"

$$= \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} \sim \text{"}\frac{-\infty}{\infty}\text{"}$$

$$\stackrel{H}{=} \lim_{x\to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{2}x^{-3/2}\right)}$$

$$= \lim_{x\to 0^+} -2x^{1/2}$$

$$= 0$$

$$\therefore \lim_{x\to 0^+} x^{\sqrt{x}} = \lim_{x\to 0^+} e^{\sqrt{x}\ln x} = e^0 = \boxed{1}$$