Question 1:

(a)[5] Use differentials to estimate the amount of paint needed to apply a coat of paint 1/2000 of a metre thick to a sphere of diameter 20 m.

Recall: the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

$$dV = \frac{4}{\cancel{3}}\pi \cancel{3}r^2\,dr$$

$$= 4\pi r^2\,dr$$

for $r = 10$, $dr = \frac{1}{2000}$:

$$dV = 4\pi(10^2)\left(\frac{1}{2000}\right)$$

$$= \boxed{\frac{\pi}{5}}$$

$\therefore$ Approximately $\frac{\pi}{5}$ m^3 of paint would be required.

(b)[5] Use a linear approximation to estimate the value of $\sqrt{e^{0.1}}$.

$f(x) = \sqrt{e^x} = e^{\frac{x}{2}}$, $a = 0$. ; $f(a) = f(0) = 1$

$f'(x) = \frac{1}{2}e^{\frac{x}{2}}$; $f'(a) = f'(0) = \frac{1}{2}$

$$L(x) = f(a) + f'(a)(x-a)$$

$$= 1 + \frac{1}{2}x$$

$\therefore$ $\sqrt{e^{0.1}} = f(0.1) \approx L(0.1) = 1 + \frac{1}{2}(0.1) = \boxed{1.05}$

Question 2:

(a)[5] Determine the inverse of the one-to-one function $y = \dfrac{1+e^x}{1-e^x}$.

$$x = \frac{1+e^y}{1-e^y}$$

$$x - xe^y = 1 + e^y$$

$$x - 1 = xe^y + e^y$$

$$x - 1 = (x+1)e^y$$

$$e^y = \frac{x-1}{x+1}$$

$$\boxed{y = \ln\left(\frac{x-1}{x+1}\right)}$$

(b)[5] Let $y = (\sin x)^{\ln x}$. Use logarithmic differentiation (or some other method) to find y'.

$$\ln y = \ln\left[(\sin x)^{\ln x}\right]$$

$$\ln y = (\ln x)\ln(\sin x)$$

$$\frac{1}{y}y' = \frac{1}{x}\ln(\sin x) + (\ln x)\frac{1}{\sin x}\cdot\cos x$$

$$\boxed{\therefore y' = (\sin x)^{\ln x}\left[\frac{1}{x}\ln(\sin x) + (\ln x)\cot x\right]}.$$

Question 3:

(a)[5] State the domain of $f(x) = \dfrac{x}{1 - \ln(x-1)}$ and determine $f'(2)$.

Domain : require (i) $x-1>0 \Rightarrow x>1$

(ii) $\ln(x-1) \neq 1 \Rightarrow x-1 \neq e$

$\Rightarrow x \neq 1+e$

$\therefore$ domain is $(1, 1+e) \cup (1+e, \infty)$.

$$f'(x) = \frac{[1-\ln(x-1)](1) - (x)\left(-\frac{1}{x-1}\right)}{[1-\ln(x-1)]^2}$$

$$f'(2) = \frac{[1-\cancel{\ln(1)}^{\,0}] - (2)\left(-\frac{1}{1}\right)}{[1-\cancel{\ln(1)}_{\,0}]^2} = \boxed{3}$$

(b)[5] Determine all critical numbers of $g(x) = \dfrac{e^x}{1+x^2}$.

$$g'(x) = \frac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2} = \frac{e^x(x^2-2x+1)}{(1+x^2)^2}$$

$$= \frac{e^x(x-1)^2}{(1+x^2)^2}$$

$g'(x)=0$? $x=1$

$g'(x)$ not exist ? no such x.

$\therefore$ $x=1$ is the only critical number.

Question 4: For this question use the function $f(x) = x^3 + \frac{3}{x}$.

(a)[5] Determine the intervals of increase and decrease of f.

$$f'(x) = 3x^2 - \frac{3}{x^2} = \frac{3x^4-3}{x^2} = \frac{3(x-1)(x+1)(x^2+1)}{x^2}$$

- $f'(x)=0$? $x = 1, -1$
- $f'(x)$ not exist ? $x = 0$

Critical numbers: -1, 0, 1

		-1		0		1	
test:	-2		$-\frac{1}{2}$		$\frac{1}{2}$		2
$f'(x) = \frac{3(x-1)(x+1)(x^2+1)}{x^2}$:	$+$		$-$		$-$		$+$
$f(x)$:	↗		↘		↘		↗

$\therefore$ f is increasing on $(-\infty, -1) \cup (1, \infty)$;
f is decreasing on $(-1, 0) \cup (0, 1)$.

(b)[5] Determine the intervals of concavity of f.

$$f''(x) = 6x + \frac{6}{x^3} = \frac{6x^4+6}{x^3} = \frac{6(x^4+1)}{x^3}$$

- $f''(x) = 0$? no such x
- $f''(x)$ not exist ? $x = 0$

		0	
test:	-1		1
$f''(x) = \frac{6(x^4+1)}{x^3}$:	$-$		$+$
$f(x)$:	⌒		⌣

$\therefore$ Graph of f is concave down on $(-\infty, 0)$,
concave up on $(0, \infty)$.

Question 5 [10 points]: Sketch the graph of a function that has all of the following properties:

1. $f(-6) = 5$, $f(0) = 8$, $f(5) = 4$
2. $\lim_{x\to 8} f(x) = -\infty$
3. $f'(x) = 0$ at $x = -6$, $x = 0$ and $x = 5$
4. $f'(x) < 0$ on $(-\infty, -6) \cup (0, 5) \cup (5, 8)$
5. $f'(x) > 0$ on $(-6, 0) \cup (8, \infty)$
6. $f''(x) > 0$ on $(-\infty, -3) \cup (3, 5)$
7. $f''(x) < 0$ on $(-3, 3) \cup (5, 8) \cup (8, \infty)$

Indicate the scale on your graph and label all inflection points.