

## Question 1:

- (a)[5] Use differentials to estimate the amount of paint needed to apply a coat of paint  $1/2000$  of a metre thick to a sphere of diameter 20 m.

Recall: the volume of a sphere of radius  $r$  is  $V = \frac{4}{3}\pi r^3$ .

$$\begin{aligned} dV &= \frac{4}{3}\pi \cdot 3r^2 dr \\ &= 4\pi r^2 dr \end{aligned}$$

for  $r = 10$ ,  $dr = \frac{1}{2000}$  :

$$\begin{aligned} dV &= 4\pi (10^2) \left(\frac{1}{2000}\right) \\ &= \boxed{\frac{\pi}{5}} \end{aligned}$$

$\therefore$  Approximately  $\frac{\pi}{5} \text{ m}^3$  of paint would be required.

- (b)[5] Use a linear approximation to estimate the value of  $\sqrt{e^{0.1}}$ .

$$f(x) = \sqrt{e^x} = e^{\frac{x}{2}}, \quad a = 0; \quad f(a) = f(0) = 1$$

$$f'(x) = \frac{1}{2}e^{\frac{x}{2}}; \quad f'(a) = f'(0) = \frac{1}{2}$$

$$L(x) = f(a) + f'(a)(x-a)$$

$$= 1 + \frac{1}{2}x$$

$$\therefore \sqrt{e^{0.1}} = f(0.1) \approx L(0.1) = 1 + \frac{1}{2}(0.1) = \boxed{1.05}$$

Question 2:

(a)[5] Determine the inverse of the one-to-one function  $y = \frac{1+e^x}{1-e^x}$ .

$$x = \frac{1+e^y}{1-e^y}$$

$$x - xe^y = 1 + e^y$$

$$x - 1 = xe^y + e^y$$

$$x - 1 = (x+1)e^y$$

$$e^y = \frac{x-1}{x+1}$$

$$y = \ln\left(\frac{x-1}{x+1}\right)$$

(b)[5] Let  $y = (\sin x)^{\ln x}$ . Use logarithmic differentiation (or some other method) to find  $y'$ .

$$\ln y = \ln\left[(\sin x)^{\ln x}\right]$$

$$\ln y = (\ln x) \ln(\sin x)$$

$$\frac{1}{y} y' = \frac{1}{x} \ln(\sin x) + (\ln x) \frac{1}{\sin x} \cdot \cos x$$

$$\therefore y' = (\sin x)^{\ln x} \left[ \frac{1}{x} \ln(\sin x) + (\ln x) \cot x \right]$$

## Question 3:

(a)[5] State the domain of  $f(x) = \frac{x}{1 - \ln(x-1)}$  and determine  $f'(2)$ .

Domain: require (i)  $x-1 > 0 \Rightarrow x > 1$

(ii)  $\ln(x-1) \neq 1 \Rightarrow x-1 \neq e$

$\Rightarrow x \neq 1+e$

$\therefore$  domain is  $(1, 1+e) \cup (1+e, \infty)$ .

$$f'(x) = \frac{[1 - \ln(x-1)](1) - (x)\left(-\frac{1}{x-1}\right)}{[1 - \ln(x-1)]^2}$$

$$f'(2) = \frac{[1 - \ln(1)] - (2)\left(-\frac{1}{1}\right)}{[1 - \ln(1)]^2} = \boxed{3}$$

(b)[5] Determine all critical numbers of  $g(x) = \frac{e^x}{1+x^2}$ .

$$\begin{aligned} g'(x) &= \frac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2} = \frac{e^x(x^2 - 2x + 1)}{(1+x^2)^2} \\ &= \frac{e^x(x-1)^2}{(1+x^2)^2} \end{aligned}$$

$$\underline{g'(x) = 0?} \quad x = 1$$

$g'(x)$  not exist? no such  $x$ .

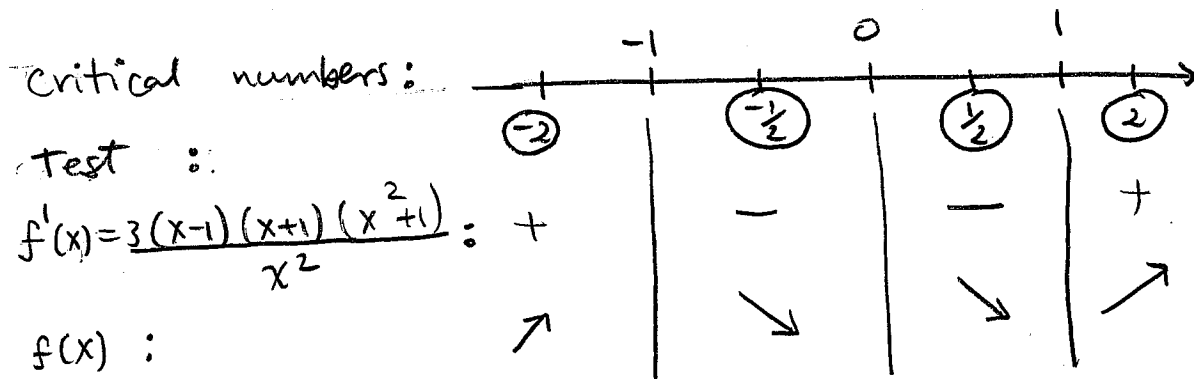
$\therefore x = 1$  is the only critical number.

Question 4: For this question use the function  $f(x) = x^3 + \frac{3}{x}$ .

(a)[5] Determine the intervals of increase and decrease of  $f$ .

$$f'(x) = 3x^2 - \frac{3}{x^2} = \frac{3x^4 - 3}{x^2} = \frac{3(x-1)(x+1)(x^2+1)}{x^2}$$

- $f'(x) = 0$  ?  $x = 1, -1$
- $f'(x)$  not exist ?  $x = 0$

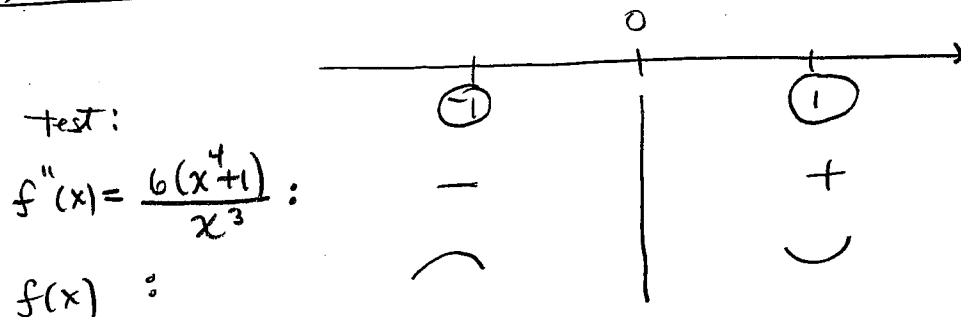


∴  $f$  is increasing on  $(-\infty, -1) \cup (1, \infty)$ ;  
 $f$  is decreasing on  $(-1, 0) \cup (0, 1)$ .

(b)[5] Determine the intervals of concavity of  $f$ .

$$f''(x) = 6x + \frac{6}{x^3} = \frac{6x^4 + 6}{x^3} = \frac{6(x^4 + 1)}{x^3}$$

- $f''(x) = 0$  ? no such  $x$
- $f''(x)$  not exist ?  $x = 0$



∴ Graph of  $f$  is concave down on  $(-\infty, 0)$ ,  
 concave up on  $(0, \infty)$ .

**Question 5 [10 points]:** Sketch the graph of a function that has all of the following properties:

1.  $f(-6) = 5$ ,  $f(0) = 8$ ,  $f(5) = 4$
2.  $\lim_{x \rightarrow 8} f(x) = -\infty$
3.  $f'(x) = 0$  at  $x = -6$ ,  $x = 0$  and  $x = 5$
4.  $f'(x) < 0$  on  $(-\infty, -6) \cup (0, 5) \cup (5, 8)$
5.  $f'(x) > 0$  on  $(-6, 0) \cup (8, \infty)$
6.  $f''(x) > 0$  on  $(-\infty, -3) \cup (3, 5)$
7.  $f''(x) < 0$  on  $(-3, 3) \cup (5, 8) \cup (8, \infty)$

Indicate the scale on your graph and label all inflection points.

