Question 1:

(a)[3 points] Let $f(x) = \dfrac{1}{1-x^2}$ and $g(x) = \sqrt{x}$. Determine, simplify, and find the domain of $(f \circ g)(x)$.

$$(f \circ g)(x) = f(g(x)) = \left.\frac{1}{1-(\sqrt{x})^2}\right\} *$$

$$= \frac{1}{1-x}$$

Using $*$, we must have $x \geq 0$ but $x \neq 1$, so domain of $f \circ g$ is $[0,1) \cup (1,\infty)$.

(b)[3 points] Let $H(x) = \sqrt[8]{2+|x|}$ and $g(x) = 2+x$. Find functions f and h so that $H = f \circ g \circ h$. (There are several possible correct answers.)

$h(x) = |x|$,

$f(x) = \sqrt[8]{x}$.

Check: $f(g(h(x))) = f(g(|x|))$

$= f(2+|x|)$

$= \sqrt[8]{2+|x|}$ ✓

(c)[4 points] Let $f(x) = \dfrac{1}{x^2}$. Evaluate and simplify $\dfrac{f(a+h)-f(a)}{h}$.

$$\frac{f(a+h)-f(a)}{h} = \frac{1}{h}\left[\frac{1}{(a+h)^2} - \frac{1}{a^2}\right]$$

$$= \frac{1}{h}\left[\frac{a^2-(a+h)^2}{(a+h)^2 a^2}\right]$$

$$= \frac{1}{h}\left[\frac{\cancel{a^2}-\cancel{a^2}-2ah-h^2}{(a+h)^2 a^2}\right]$$

$$= \boxed{\frac{-2a-h}{(a+h)^2 a^2}}$$

Question 2:

(a)[3 points] Evaluate: $\lim_{x \to -3} \frac{x^2 - \sqrt{x^2-9}}{x-3}$

$$= \frac{9 - \sqrt{9-9}}{-6}$$

$$= \boxed{\frac{-3}{2}}$$

(b)[3 points] Evaluate: $\lim_{x \to 4} \frac{x^2 - x - 12}{x^2 - 7x + 12}$ $\sim$ "$\frac{0}{0}$"

$$= \lim_{x \to 4} \frac{\cancel{(x-4)}(x+3)}{\cancel{(x-4)}(x-3)}$$

$$= \boxed{7}$$

(c)[4 points] Evaluate: $\lim_{x \to 0} \frac{x}{\sqrt{1+3x}-1}$ $\sim$ "$\frac{0}{0}$"

$$= \lim_{x \to 0} \frac{x}{\sqrt{1+3x}-1} \cdot \frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1}$$

$$= \lim_{x \to 0} \frac{x(\sqrt{1+3x}+1)}{\cancel{1}+3x-\cancel{1}}$$

$$= \lim_{x \to 0} \frac{\cancel{x}(\sqrt{1+3x}+1)}{3\cancel{x}} = \boxed{\frac{2}{3}}$$

Question 3:

(a)[5 points] Evaluate: $\lim_{\theta\to 0} \frac{3\sin\theta}{2\theta - 5\tan\theta}$ $\sim$ "$\frac{0}{0}$"

$$\lim_{\theta\to 0} \frac{3\sin\theta}{2\theta - 5\tan\theta} = \lim_{\theta\to 0} \frac{3\sin\theta}{2\theta - 5\frac{\sin\theta}{\cos\theta}} \begin{matrix} \div \theta \\ \div \theta \end{matrix}$$

$$= \lim_{\theta\to 0} \frac{\left(3\frac{\sin\theta}{\theta}\right)}{\left(\frac{2\cancel{\theta}}{\cancel{\theta}} - 5\frac{\sin\theta}{\theta}\cdot\frac{1}{\cos\theta}\right)}$$

$$= \frac{3}{2-5}$$

$$= \boxed{-1}$$

(b)[5 points] Evaluate: $\lim_{t\to 0}\left(\frac{1}{t} - \frac{1}{t^2+t}\right) = \lim_{t\to 0}\left(\frac{1}{t} - \frac{1}{t(t+1)}\right)$

$$= \lim_{t\to 0} \frac{t+1-1}{t(t+1)}$$

$$= \lim_{t\to 0} \frac{\cancel{t}}{\cancel{t}(t+1)}$$

$$= \boxed{1}$$

Question 4:

(a)[3 points] If $2x \le g(x) \le x^4 - x^2 + 2$ for every x, evaluate $\lim_{x\to 1} g(x)$. Be sure to state any theorems you use (the Squeeze Theorem, for example), and the conditions that are satisfied which justify use of the theorem.

Observe: $\lim_{x\to 1} 2x = 2 = \lim_{x\to 1} x^4 - x^2 + 2$

Since $2x \leqslant g(x) \leqslant x^4 - x^2 + 2$, by the Squeeze Theorem, $\boxed{\lim_{x\to 1} g(x) = 2}$

(b)[3 points] Evaluate: $\lim_{x\to 2^+} \dfrac{\cos(\pi x)}{2-x}$

As $x \to 2^+$, $\cos(\pi x) \to 1$ while $2 - x \to 0^-$

$\therefore \lim_{x\to 2^+} \dfrac{\cos(\pi x)}{2-x} = \boxed{-\infty}$

(c)[4 points] Evaluate: $\lim_{x\to 2} \dfrac{\cos(\pi x)}{2-x}$

From (b) we have $\lim_{x\to 2^+} \dfrac{\cos(\pi x)}{2-x} = +\infty$.

Now consider $\lim_{x\to 2^-} \dfrac{\cos(\pi x)}{2-x}$: as $x \to 2^-$, again $\cos(\pi x) \to 1$, yet $2 - x \to 0^+$, so $\lim_{x\to 2^-} \dfrac{\cos(\pi x)}{2-x} = -\infty$.

Since the left and right hand limits differ, $\lim_{x\to 2} \dfrac{\cos(\pi x)}{2-x}$ $\boxed{\text{does not exist}}$

Question 5:

(a)[5 points] Use the Intermediate Value Theorem to show that the equation

$$\cos x = 3 - 2x$$

has at least one real root on the interval $[0, \pi]$.

$\cos x = 3 - 2x$

$\cos x - 3 + 2x = 0$

Let $f(x) = \cos x - 3 + 2x$, a continuous function.

$f(0) = \underbrace{\cos(0)}_{1} - 3 + \underbrace{2(0)}_{0} = -2$

$f(\pi) = \underbrace{\cos(\pi)}_{-1} - 3 + 2\pi = -4 + 2\pi > 0$

Since $f(0) < 0 < f(\pi)$, there is some number $0 < c < \pi$ such that $f(c) = 0$.

(b)[5 points] Evaluate: $\displaystyle \lim_{x\to-\infty} \frac{6x^7 - 7x^5 - 5}{7x^6 - 6x^5 + 5} \begin{matrix} \div x^6 \\ \div x^6 \end{matrix}$

$$= \lim_{x \to -\infty} \frac{6x - \frac{7}{x} - \frac{5}{x^6}}{7 - \frac{6}{x} + \frac{5}{x^6}}$$

$$= \boxed{-\infty}$$