Question 1 [12 points]: Evaluate the following limits (you may use the fact that $\lim_{x\to 0} \frac{\sin x}{x} = 1$):

(a) $\lim_{x\to\infty} \frac{2e^x - 7e^{-x} + 3}{5e^x + 4e^{-x} - 1}$ $= \lim_{x\to\infty} \frac{2 - 7e^{-2x} + 3e^{-x}}{5 + 4e^{-2x} - e^{-x}}$

$= \frac{2}{5}$

(b) $\lim_{x\to 5} \frac{x^2 - 7x + 10}{x^2 - 8x + 15}$ $= \lim_{x\to 5} \frac{(x-5)(x-2)}{(x-5)(x-3)} = \frac{3}{2}$

(c) $\lim_{x\to 0} \frac{\tan(3x)\cos(2x)}{x}$ $= \lim_{x\to 0} \frac{\sin(3x)}{3x} \cdot \frac{3}{\cos(3x)} \cdot \cos(2x)$

$= 3$

Question 2 [8 points]:

(a) Let $f(x) = \sqrt{x+1}$. Use the <u>definition of the derivative</u> to find $f'(x)$. (No credit will be given if $f'(x)$ is found using differentiation rules.)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\sqrt{x+h+1} - \sqrt{x+1}\right) \cdot \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + x+1}$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}} \cdot \frac{\cancel{x}+\cancel{h}+\cancel{1}-\cancel{x}-\cancel{1}}{\sqrt{x+h+1} + \sqrt{x+1}}$$

$$= \frac{1}{2\sqrt{x+1}}$$

(b) Find the linear approximation to $f(x) = \sqrt{x+1}$ at $a = 3$.

$$f(x) = \sqrt{x+1} \; ; \quad f(3) = \sqrt{4} = 2$$

$$f'(x) = \frac{1}{2\sqrt{x+1}} \; ; \quad f'(3) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$\therefore \; L(x) = f(a) + f'(a)(x-a)$$

$$= 2 + \frac{1}{4}(x-3).$$

$$= \frac{1}{4}x + \frac{5}{4}$$

Question 3 [15 points]: Differentiate the following functions (you do not need to simplify your answers):

(a) $y = 5\cos(x^7 - 8x)$

$y' = -5\sin(x^7 - 8x) \cdot (7x^6 - 8)$

(b) $y = e^x \cot x$

$y' = e^x \cot x - e^x \csc^2 x$

(c) $y = \dfrac{2\sqrt{x}}{e^x - \pi x}$ (Use the Quotient Rule.)

$$y' = \frac{(e^x - \pi x)\, 2 \cdot \frac{1}{2} x^{-\frac{1}{2}} - 2\sqrt{x}\,(e^x - \pi)}{(e^x - \pi x)^2}$$

$$= \frac{(e^x - \pi x)\, x^{-1/2} - 2\, x^{1/2}\,(e^x - \pi)}{(e^x - \pi x)^2}$$

(d) $y = x^3 4^{\tan x}$

$y' = 3x^2 4^{\tan x} + x^3 4^{\tan x} \cdot \ln 4 \cdot \sec^2 x$

(e) $y = \sqrt[5]{\log_7(x^2 - \cos x)}$ $= \left[\log_7(x^2 - \cos x)\right]^{\frac{1}{5}}$

$$y' = \frac{1}{5}\left[\log_7(x^2 - \cos x)\right]^{-\frac{4}{5}} \cdot \frac{1}{(x^2 - \cos x)\ln 7} \cdot (2x + \sin x).$$

Question 4 [10 points]:

(a) Find the general antiderivative of the following functions:

(i) $f(x) = \sec^2 x + 3\sin x - \frac{e^x}{2}$

$F(x) = \tan x - 3\cos x - \frac{1}{2}e^x + C$

(ii) $y = \frac{2x^4 - \sqrt[3]{x^2} + 7}{x}$ $= 2x^3 - x^{-1/3} + 7x^{-1}$

$\therefore F(x) = \frac{2x^4}{4} - \frac{x^{2/3}}{2/3} + 7\ln|x| + C$

$= \frac{x^4}{2} - \frac{3}{2}x^{2/3} + 7\ln|x| + C$

(b) Find the function $f(t)$ with $f'(t) = 4t^3 + 2t$ and $f(1) = 9$.

$f(t) = \frac{4t^4}{4} + \frac{2t^2}{2} + C = t^4 + t^2 + C$

$f(1) = 9 \Rightarrow 1 + 1 + C = 9$

$\therefore C = 7$

$\therefore f(t) = t^4 + t^2 + 7.$

Question 5 [10 points]:

(a) Find $\frac{dy}{dx}$ by implicit differentiation:

$$xy + e^{3x} = \sin(x+y)$$

$$y + xy' + 3e^{3x} = \cos(x+y)(1+y')$$

$$y + xy' + 3e^{3x} = \cos(x+y) + y'\cos(x+y)$$

$$y'(x - \cos(x+y)) = \cos(x+y) - y - 3e^{3x}$$

$$\therefore \quad y' = \frac{\cos(x+y) - y - 3e^{3x}}{x - \cos(x+y)}$$

(b) Use logarithmic differentiation to find $\frac{dy}{dx}$:

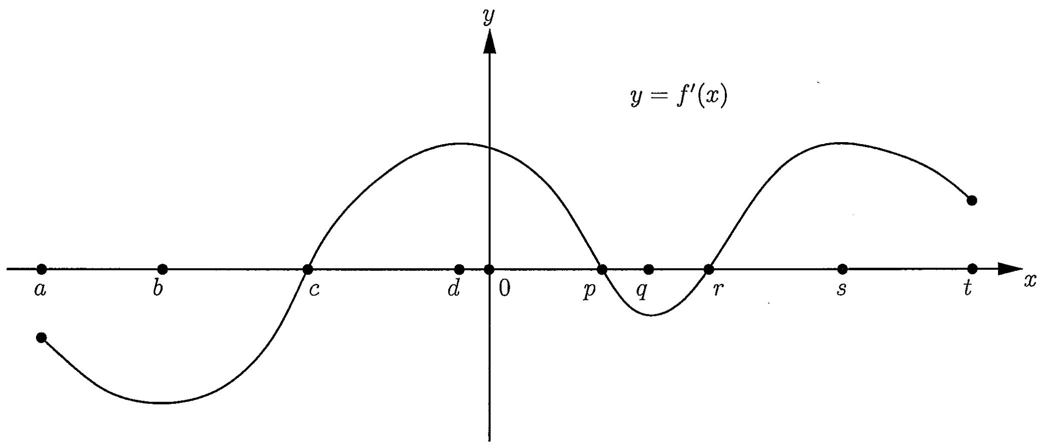
$$y = \frac{3^{\sin x}}{(2x+1)^{10}}$$

$$\ln y = \sin x \cdot \ln 3 - 10\ln(2x+1)$$

$$\frac{1}{y}y' = \cos x \cdot \ln 3 - 10\frac{1}{2x+1} \cdot 2$$

$$\therefore \; y' = \left(\cos x \cdot \ln 3 - \frac{20}{2x+1}\right)\frac{3^{\sin x}}{(2x+1)^{10}}$$

Question 6 [8 points]: Consider the graph of $f'(x)$ below (note this is the graph of $f'(x)$, not $f(x)$):



(a) On what interval(s) is f increasing?

(c, p) & (r, t)

(b) At what value(s) of x does f have local minima?

$x = c$, $x = r$

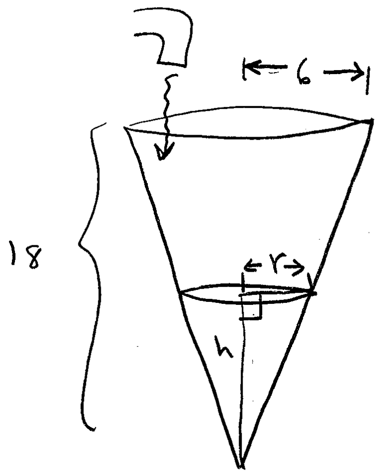
(c) On what interval(s) is the graph of f concave down?

(a, b) , (d, q) , (s, t)

(d) At what value(s) of x does the graph of f have points of inflection?

$x = b$, $x = d$, $x = q$, $x = s$

Question 7 [10 points]: A water tank has the shape of an inverted circular cone with top radius 6 metres and height 18 metres. Water is pumped into the tank at a rate of 10 cubic metres per minute. How fast is the water level rising when the water is 2 metres deep? (Recall, the volume of of a right circular cone of height h and radius r is $V = \frac{1}{3}\pi r^2 h$.)



$$\frac{dV}{dt} = 10 \frac{m^3}{min.}$$

Find $\frac{dh}{dt}$ when $h = 2$ m.

$$V = \frac{1}{3}\pi r^2 h.$$

$$\frac{h}{18} = \frac{r}{6}$$

$$\therefore r = \frac{1}{3}h$$

$$\therefore V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h$$

$$= \frac{1}{27}\pi h^3$$

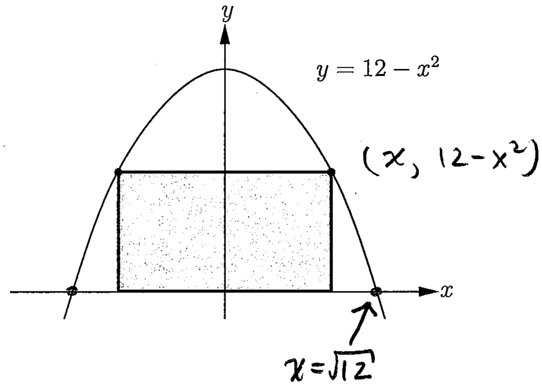
$$\frac{dV}{dt} = \frac{\pi}{27} 3h^2 \frac{dh}{dt}$$

$\therefore$ when $h = 2$ m:

$$10 = \frac{\pi}{27} 3 \cdot 2^2 \cdot \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{27 \cdot 10}{3 \cdot 2^2 \pi} = \frac{45}{2\pi} \frac{m}{min}.$$

Question 8 [10 points]: A rectangle with base on the x-axis has its upper vertices on the curve $y = 12 - x^2$ (see figure below.) Find the maximum area of such a rectangle.



$\therefore \; A(x) = 2x(12-x^2).$

Maximize $A(x)$ on $[0, \sqrt{12}]$.

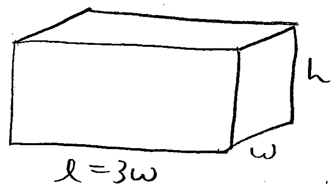
$$A'(x) = 2(12-x^2) + 2x(-2x)$$
$$= 24 - 6x^2$$
$$= 6[4 - x^2]$$
$$= 6(2-x)(2+x)$$

$A'(x) = 0$ at $x = 2, -2$.

x	$A(x)$
0	0
2	$4(12-4) = 32$
$\sqrt{12}$	0

$\therefore$ The maximum area is 32 units^2.

Question 9 [10 points]: A rectangular box without a top is constructed from cardboard. The length of the box is equal to three times the width. The volume of the box is 18 cubic metres. Find the <u>dimensions</u> of the box which minimize the amount of cardboard used.



Let S = surface area of box.

$V = 18 \text{ m}^3$

$\therefore \; lwh = 18$

$(3w)wh = 18$

$3w^2h = 18$

$\therefore \; S = lw + 2lh + 2wh$

$= 3w^2 + 6wh + 2wh$

$= 3w^2 + 8wh$

Minimize $S = 3w^2 + 8wh$ subject to $3w^2h = 18$.

$3w^2h = 18 \Rightarrow h = \frac{18}{3w^2} = \frac{6}{w^2}$.

$\therefore \; S(w) = 3w^2 + 8w\left(\frac{6}{w^2}\right) = 3w^2 + \frac{48}{w}$, $w > 0$

$S'(w) = 6w - \frac{48}{w^2} = \frac{6w^3 - 48}{w^2}$

$S'(w) = 0$ at $w = 2$.

	0	①	2	③
$S'(w)$:	(	$-$	0	$+$
$S(w)$:		$\searrow$		$\nearrow$

$\therefore$ $w = 2$ yields the absolute minimum of S.

$\therefore$ $w = 2$, $h = \frac{6}{w^2} = \frac{6}{4} = \frac{3}{2}$, $l = \frac{18}{wh} = \frac{18}{(2)(\frac{3}{2})} = 6$

i.e. $\boxed{w = 2,\ h = \frac{3}{2}\ \&\ l = 6 \text{ m}}$

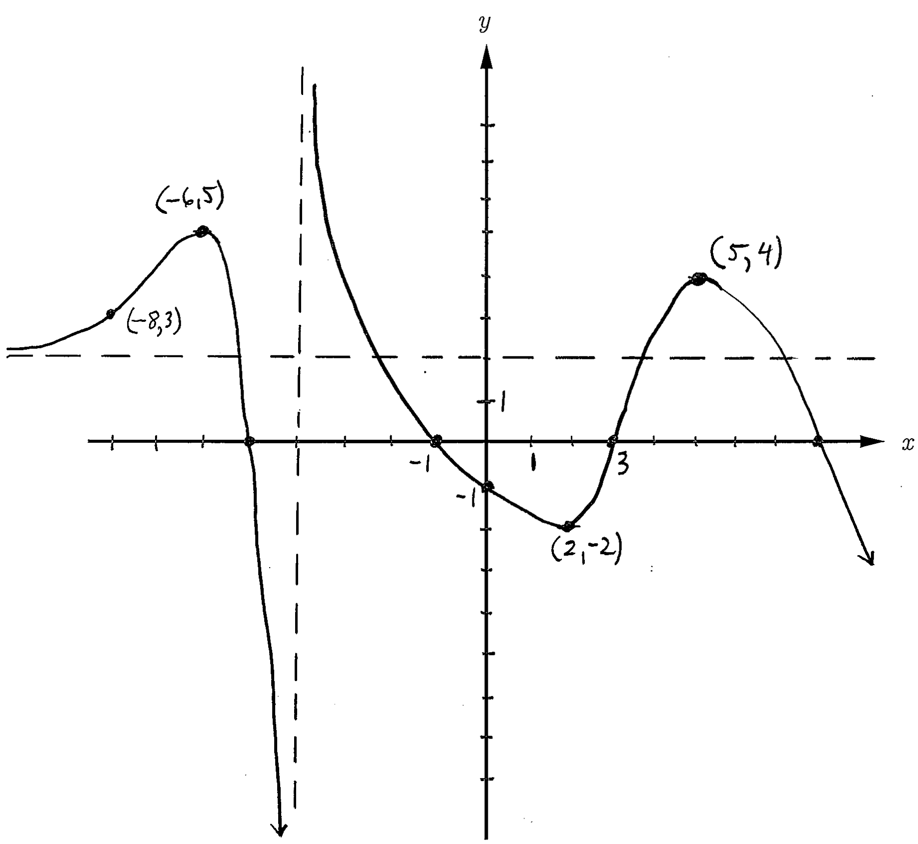
Question 10 [5 points]: Suppose you have analyzed a function and found the following:

1. the domain of f is $(-\infty, -4)$, $(-4, \infty)$
2. f has the following function values:

x	-8	-6	-5	-1	0	2	3	5	8
$f(x)$	3	5	0	0	-1	-2	0	4	0

3. $\lim_{x\to-\infty} f(x) = 2$, $\quad \lim_{x\to\infty} f(x) = -\infty$
4. $\lim_{x\to-4^+} f(x) = \infty$, $\quad \lim_{x\to-4^-} f(x) = -\infty$
5. $f'(-6) = f'(2) = f'(5) = 0$
6. $f'(x) > 0$ on $(-\infty, -6)$ and $(2, 5)$
7. $f'(x) < 0$ on $(-6, -4)$, $(-4, 2)$ and $(5, \infty)$
8. $f''(-8) = f''(3) = 0$
9. $f''(x) > 0$ on $(-\infty, -8)$ and $(-4, 3)$
10. $f''(x) < 0$ on $(-8, -4)$ and $(3, \infty)$

Neatly sketch the graph of $y = f(x)$.



Question 11 [12 points]: The function $f(x) = x^4 e^{-x}$ has first derivative $f'(x) = (4x^3 - x^4)e^{-x}$ and second derivative $f''(x) = (x^4 - 8x^3 + 12x^2)e^{-x}$.

(a) Find the intervals on which $f(x)$ is increasing or decreasing.

$$f'(x) = 0 \Rightarrow 4x^3 - x^4 = 0$$
$$\Rightarrow x^3(4 - x) = 0$$
$$\Rightarrow x = 0, 4$$

	-1	0	1	4	5
$f'(x)$:	$-$	0	$+$	0	$-$
$f(x)$:	$\searrow$	0	$\nearrow$	$256e^{-4}$	$\searrow$

$\therefore$ f increasing on $(0,4)$, decreasing on $(-\infty,0) \cup (4,\infty)$.

(b) Find the local maximum and minimum values of $f(x)$.

f has a local minimum of 0 at $x = 0$.

f has a local maximum of $256e^{-4}$ at $x = 4$.

(c) Find the intervals on which $f(x)$ is concave up or concave down.

$$f''(x) = 0 \Rightarrow x^4 - 8x^3 + 12x^2 = 0$$
$$x^2(x^2 - 8x + 12) = 0$$
$$x^2(x-2)(x-6) = 0$$
$$x = 0, 2, 6$$

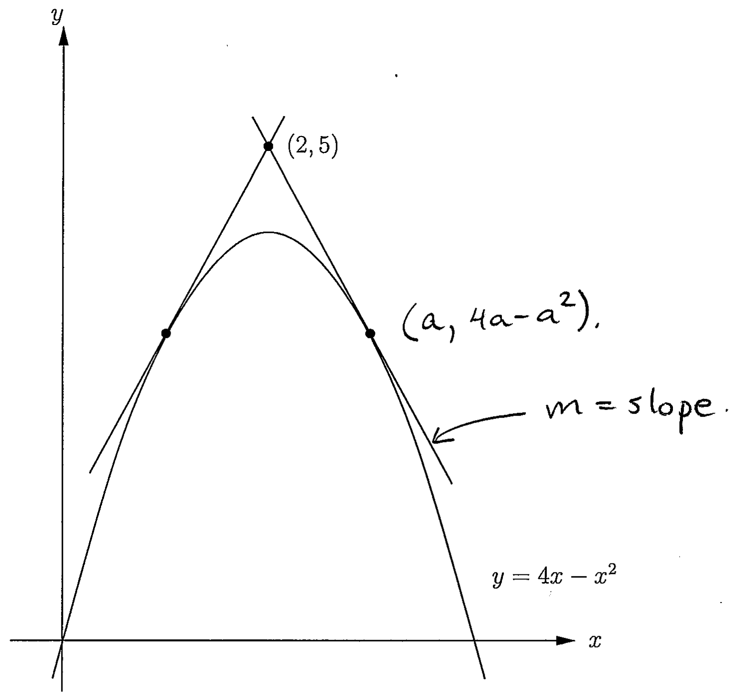
	-1	0	1	2	3	6	7
$f''(x)$:	$+$	0	$+$	0	$-$	0	$+$
$f(x)$:	C.U.	0	C.U	$16e^{-2}$	C.d.	$1296e^{-6}$	C.U.

$\therefore$ f is C.U. on $(-\infty,0) \cup (0,2) \cup (6,\infty)$; C.d. on $(2,6)$.

(d) Find the inflection points of $f(x)$.

$(2, 16e^{-2})$, $(6, 1296e^{-6})$.

Question 12 [5 points]: There are two tangent lines to the curve $y = 4x - x^2$ that pass through the point $(2, 5)$; see the diagram below. Find the coordinates of the points where these tangent lines intersect the parabola.



$$m = \left.\frac{dy}{dx}\right|_{x=a} = \left.4-2x\right|_{x=a} = 4-2a.$$

also, $m = \dfrac{4a-a^2-5}{a-2}$

$\therefore \quad \dfrac{4a-a^2-5}{a-2} = 4-2a$

$4a-a^2-5 = (4-2a)(a-2)$

$\cancel{4a}-a^2-5 = \cancel{4a} - 2a^2 - 8 + 4a$

$a^2 - 4a + 3 = 0$

$(a-3)(a-1) = 0$

$\therefore \quad a = 1, 3$

$\therefore$ points are $(1,3)$ & $(3,3)$.

$a = 1 \Rightarrow 4a - a^2 = 3$

$a = 3 \Rightarrow 4a - a^2 = 3$

Question 13 [5 points]: Suppose $f(x) = axe^{bx}$ where a and b are constants. If $f(1/3) = 1$ and $y = f(x)$ has a maximum at $x = 1/3$, find the values of a and b.

$$f(\tfrac{1}{3}) = 1 \Rightarrow a(\tfrac{1}{3})e^{\frac{1}{3}b} = 1 \quad \cdots (1)$$

$$f'(\tfrac{1}{3}) = 0 \Rightarrow ae^{bx} + axe^{bx} \cdot b \Big|_{x=\frac{1}{3}} = 0$$

$$\Rightarrow ae^{\frac{1}{3}b} + a \cdot \tfrac{1}{3} \cdot e^{\frac{1}{3}b} \cdot b = 0$$

$$\Rightarrow ae^{\frac{1}{3}b}\left(1 + \tfrac{b}{3}\right) = 0 \quad \cdots (2)$$

$\therefore$ $a = 0$ or $b = -3$,

but $a \neq 0$ because of (1), so $b = -3$.

$$\therefore (1) \Rightarrow a(\tfrac{1}{3})e^{\frac{1}{3}(-3)} = 1$$

$$\frac{a}{3}e^{-1} = 1$$

$$a = 3e$$

$$\boxed{\therefore \; a = 3e, \quad b = -3}$$