1 First Derivatives and Shapes of Curves

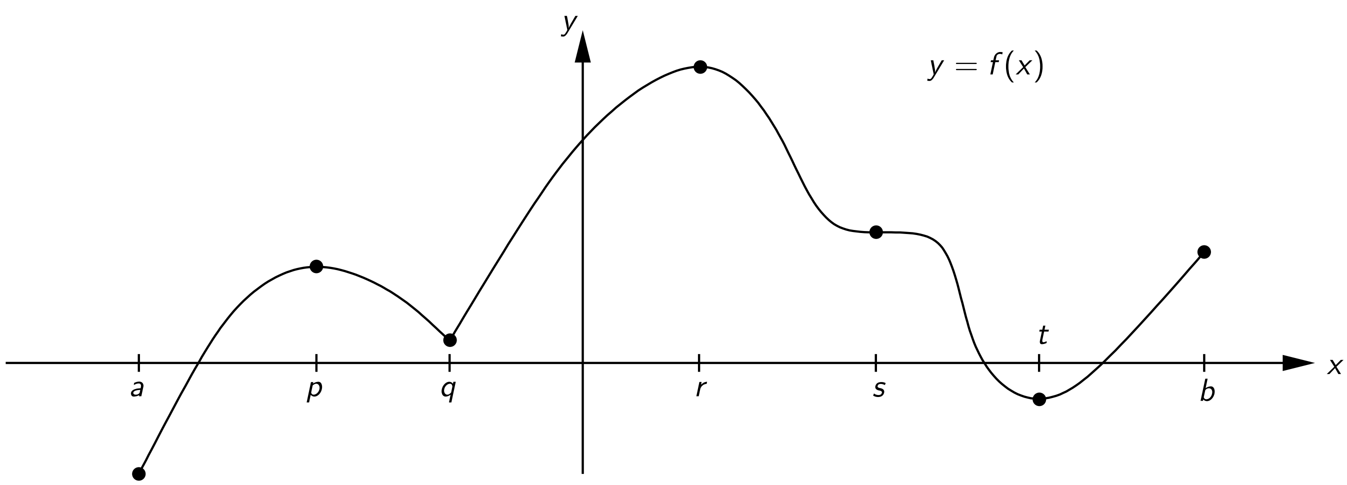
What information does $f'(x)$ give us about the shape of the graph of $y = f(x)$? In particular, we wish to determine

(i) the intervals over which the outputs of f are increasing, i.e. the intervals over which the graph of $y = f(x)$ is rising

(ii) the intervals over which the outputs of f are decreasing, i.e. the intervals over which the graph of $y = f(x)$ is falling

(iii) the peaks and valleys between these intervals

Using the following general graph let's introduce some terminology and make some observations:



Here f has domain $D = [a, b]$.

1.1 Definitions and a Theorem

absolute (or global) maximum: f has an absolute maximum of $f(c)$ at $x = c$ if $f(c) \geq f(x)$ for every x in D.

absolute (or global) minimum: f has an absolute minimum of $f(c)$ at $x = c$ if $f(c) \leq f(x)$ for every x in D.

extreme values of f: the absolute maximum of f together with the absolute minimum.

relative (or local) maximum: f has a relative maximum of $f(c)$ at $x = c$ if $f(c) \geq f(x)$ for every x in an open interval containing c.

relative (or local) minimum: f has a relative minimum of $f(c)$ at $x = c$ if $f(c) \leq f(x)$ for every x in an open interval containing c.

So, referring to the graph above, we would say:

- f has an absolute maximum of $f(r)$ at $x = r$;
- f has an absolute minimum of $f(a)$ at $x = a$;
- f has relative maxima of $f(p)$ at $x = p$ and $f(r)$ at $x = r$;
- f has a relative minima of $f(q)$ at $x = q$ and $f(t)$ at $x = t$

Note:

(i) End points can correspond to absolute but not relative maxima or minima.

(ii) A point interior to the interval can correspond to both a relative and absolute maximum or minimum.

Another definition:

critical number: a critical number of a function f is a number c in the domain of f such that

(i) $f'(c) = 0$, or
(ii) $f'(c)$ does not exist

Referring to our graph, $x = p$, $x = q$, $x = r$, $x = s$ and $x = t$ are critical numbers of f. Notice the behaviour of the graph of $y = f(x)$ at each of these critical numbers. Indeed,

Fermat's Theorem: If f has a relative maximum or relative minimum at $x = c$ and if $f'(c)$ exists, then $f'(c) = 0$.

Fermat's Theorem tells us that relative extrema must occur at critical numbers, however it does not say that every critical number corresponds to a relative extremum—look at $x = s$ in our graph above.

1.2 Increasing/Decreasing Test

Recall

$$f'(c) = \text{slope of the tangent line to graph of } y = f(x) \text{ at } x = c\ ,$$

so

$$\begin{aligned} f'(c) > 0 &\Rightarrow \text{outputs of } f \text{ are increasing as } x \text{ passes through } c \\ &\Rightarrow \text{graph of } f \text{ is rising as } x \text{ passes through } c \\ f'(c) < 0 &\Rightarrow \text{outputs of } f \text{ as } x \text{ passes through } c \\ &\Rightarrow \text{graph of } f \text{ is falling as } x \text{ passes through } c \end{aligned}$$

This gives the **Test for Intervals of Increase and Decrease of a Function:**

(i) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(ii) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Observe on our graph: whenever f changes from increasing to decreasing, or vice versa, it does so at a critical number ($x = p, q, r$ and t). However, not every critical number corresponds to such a change: f is decreasing on both sides of $x = s$. Putting all of this together:

To determine the intervals of increase and decrease of a function f:

(i) Find points at which f' changes sign (from positive to negative or vice versa). f' can change sign at

- critical numbers: x-values at which $f'(x) = 0$ or $f'(x)$ does not exist
- values of x at which f itself is not defined

(ii) Test $f'(x)$ on the subintervals defined by the points from (i).

Once you have determined the intervals of increase/decrease of f, it is easy to read off the relative extrema (that is, the relative maxima and minima) using

The First Derivative Test: Suppose $x = c$ is a critical number of a continuous function f.

(i) If f' changes from positive to negative at $x = c$, then f has a relative maximum of $f(c)$ at $x = c$.

(ii) If f' changes from negative to positive at $x = c$, then f has a relative minimum of $f(c)$ at $x = c$.

(iii) If f' does not change sign at $x = c$, then f has a neither a relative maximum nor relative minimum at $x = c$.

Example 1

Let $f(x) = 3x^{2/3} - x$.

(i) Determine the intervals of increase and decrease of f.

(ii) State the relative extrema.

(iii) Use the information from (i) and (ii) to sketch the graph of $y = f(x)$.

Example 2

Let $f(t) = t + \cos(t)$ where $-2\pi \leq t \leq 2\pi$.

(i) Determine the intervals of increase and decrease of f.

(ii) State the relative extrema.

(iii) Use the information from (i) and (ii) to sketch the graph of $y = f(t)$.