Question 1 [10 points]: For this question you may state equations of lines using any of the standard forms.

(a)[4] Determine an equation of the line passing through the points $(-7, 3)$ and $(-2, -3)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-3}{-2-(-7)} = \frac{-6}{5}$$

$$\therefore\ y - 3 = \frac{-6}{5}(x - (-7))$$

$$\boxed{y - 3 = \frac{-6}{5}(x+7)}$$

or $\boxed{y = \frac{-6}{5}x - \frac{27}{5}}$

(b)[3] Determine an equation of the vertical line through the point $(11, -11)$.

$$\boxed{x = 11}$$

(c)[3] Determine an equation of the line which is parallel to the line $3x - 5y = 12$ and which has y-intercept $(0, 3)$.

$$3x - 5y = 12 \implies -5y = -3x + 12$$

$$y = \frac{3}{5}x - \frac{12}{5}$$

$$\therefore\ m = \frac{3}{5}$$

$$\therefore\ \boxed{y - 3 = \frac{3}{5}(x - 0)}$$

or $\boxed{y = \frac{3}{5}x + 3}$

Question 2 [10 points]:

(a)[5] A doughnut shop sells doughnuts for \$3.29 per dozen. The shop has fixed weekly costs of \$650 and it costs \$1.55 to make a dozen doughnuts. How many dozen doughnuts must be sold each week in order for the business to break even?

x = number of dozen sold.

$R = 3.29x$

$C = 650 + 1.55x.$

$R = C \Rightarrow 3.29x = 650 + 1.55x$

$1.74x = 650$

$x = \frac{650}{1.74}$

$x \doteq 374.$

$\therefore$ 374 dozen doughnuts must be sold each week to break even.

(b)[5] A company has data indicating that when the price of a particular product is \$138 the quantity demanded is 72 while the quantity supplied is 96. At the market (or equilibrium) price of \$120 the quantity demanded increases to 88. Determine the supply equation for the product.

$(138, 72)$ is a point on the demand line.
$(138, 96)$ is a point on the supply line.
$(120, 88)$ is a point on the demand line.
\$120 is market price, so $(120, 88)$ is also on the supply line.
$\therefore$ Supply equation is that of line through $(138, 96)$ & $(120, 88)$.

$m = \frac{96-88}{138-120} = \frac{8}{18} = \frac{4}{9}$

$\therefore \boxed{S - 88 = \frac{4}{9}(p - 120)}$ or $\boxed{S = \frac{4}{9}p + \frac{104}{3}}$

Question 3 [10 points]:

(a)[5] A local charity sells boxes of oranges and boxes of grapefruit to raise money. A box of oranges costs \$14 and a box of grapefruit costs \$16. The fundraising drive ended up raising \$7570 from the sale of 502 boxes of fruit. How many boxes of each type of fruit were sold?

Let x = number of boxes of oranges sold.
y = " " " " grapefruit " .

① $x+y=502$
② $14x+16y=7570$

① $\Rightarrow y=502-x$
② $\Rightarrow 14x+16(502-x)=7570$

$$14x+8032-16x=7570$$
$$-2x=-462$$
$$x=231$$
$$\therefore y=502-231=271$$

∴ 231 boxes of oranges and
271 boxes of grapefruit were sold.

(b)[5] An investor has \$12,000 to invest and two investments are available: one pays 4% simple interest per year, while the second riskier investment pays 6.5% simple interest per year. The investor has a goal of earning \$570 for the year. How much should be invested in each of the investments?

Let x = amount invested at 4%
y = amount invested at 6.5%.

① $x+y=12\,000$
② $0.04x+0.065y=570$

① $\Rightarrow y=12000-x$
② $\Rightarrow 0.04x+0.065(12000-x)=570$

$$-0.025x+780=570$$
$$x=\frac{-210}{-0.025}=\$8400$$
$$\therefore y=12000-8400=\$3600$$

∴ \$8400 should be invested at 4%,
3600 " " " " 6.5%.

Question 4 [10 points]: Solve the following system of equations **using matrix reduction** (no credit will be given for using any other method). Use proper notation to clearly state the row operations used at each step and clearly state the final solution.

$$-x + 2y + z = 1$$
$$-5x + 8y + 2z = 3$$
$$7x - 11y - 3z = -2$$

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 1 \\ -5 & 8 & 2 & 3 \\ 7 & -11 & -3 & -2 \end{array}\right]$$

$R_1 = (-1) r_1$:

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & -1 \\ -5 & 8 & 2 & 3 \\ 7 & -11 & -3 & -2 \end{array}\right]$$

$R_2 = (5) r_1 + r_2$:

$R_3 = (-7) r_1 + r_3$;

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & -1 \\ 0 & -2 & -3 & -2 \\ 0 & 3 & 4 & 5 \end{array}\right]$$

$R_2 = r_3 + r_2$:

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 3 & 4 & 5 \end{array}\right]$$

$R_3 = (-3) r_2 + r_3$:

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

$\therefore z = -4$;

$y + z = 3 \Rightarrow y = 3 - z = 3 + 4 = 7$;

$x - 2y - z = -1$

$\Rightarrow x = -1 + 2y + z$

$= -1 + 2(7) + (-4)$

$= 9$

$$\boxed{\therefore x = 9, \quad y = 7, \quad z = -4}$$

Question 5 [10 points]: For this problem use the following matrices:

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 3 & -1 \\ 2 & 4 \\ 1 & 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$$

(a)[3] Compute $\mathbf{AB} - 4\mathbf{C}$.

$$AB - 4C = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 4 \\ 1 & 0 \end{bmatrix} - 4\begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -3 \\ 10 & 2 \end{bmatrix} + \begin{bmatrix} 4 & -4 \\ -12 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -7 \\ -2 & 10 \end{bmatrix}$$

(b)[4] Determine $\mathbf{C}^{-1}$.

$$\left[\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 3 & -2 & 0 & 1 \end{array}\right]$$

$R_1 = (-1) r_1$:

$$\left[\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 3 & -2 & 0 & 1 \end{array}\right]$$

$R_2 = (-3) r_1 + r_2$:

$$\left[\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 0 & 1 & 3 & 1 \end{array}\right]$$

$R_1 = (1) r_2 + r_1$:

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 1 \end{array}\right]$$

$$\therefore C^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$$

(c)[3] Solve

$$\mathbf{C}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for x and y. Your answer to part (b) should make this easy.

$$\begin{bmatrix} x \\ y \end{bmatrix} = C^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

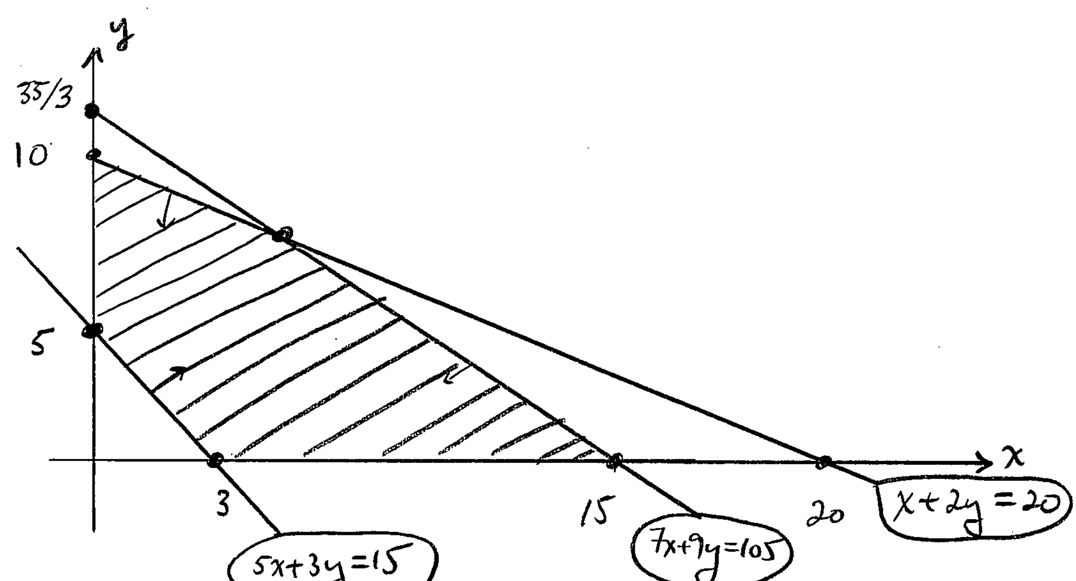
$\therefore$ $x = 3$, $y = 4$

Question 6 [10 points]: Maximize $z = 4x + 6y$ subject to the constraints

$$5x + 3y \geq 15$$
$$x + 2y \leq 20$$
$$7x + 9y \leq 105$$
$$x \geq 0$$
$$y \geq 0$$

Keep your work organized: neatly draw any required graphs and clearly show your work when determining corner points. State a clear conclusion.

Inequality	equation	test pt.	result
$5x+3y \geq 15$	$5x+3y=15$	$(0,0)$	false
$x+2y \leq 20$	$x+2y=20$	$(0,0)$	true.
$7x+9y \leq 105$	$7x+9y=105$	$(0,0)$	true.



Cornerpts: (i) by inspection: $(0,10), (0,5), (3,0), (15,0)$.

(ii) Solving ① $7x+9y=105$, ② $x+2y=20$

② $\Rightarrow x = 20-2y$

① $\Rightarrow 7(20-2y)+9y=105$

$-5y = -35$

$y = 7$

$x = 6$.

$\therefore (6,7)$.

Corner pts	$z=4x+6y$
$(0,10)$	60
$(0,5)$	30
$(3,0)$	12
$(15,0)$	60
$(6,7)$	66 ← max.

$\therefore$ z has a maximum of 66 at $x=6, y=7$

Question 7 [10 points]:

(a)[3] What effective rate of interest is equivalent to 5.5% compounded monthly?

Solve $1+R = \left(1+\frac{0.055}{12}\right)^{12}$

$$\Rightarrow R = \left(1+\frac{0.055}{12}\right)^{12}-1$$

$$R \doteq 0.0564$$

$$\boxed{R \doteq 5.64\,\%}$$

(b)[3] A deposit made today has a value of $10,000 at the end of 10 years. If the interest rate is 5.6% compounded quarterly what was the original deposit amount?

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$ with $A = 10{,}000$, $r = 0.056$, $n = 4$, $t = 10$

$$\therefore P = \frac{A}{\left(1+\frac{r}{n}\right)^{nt}}$$

$$P = \frac{10\,000}{\left(1+\frac{0.056}{4}\right)^{(4)(10)}} \doteq \boxed{\$5734.32}$$

(c)[4] An investment triples in 12 years. What rate of interest compounded semiannually will achieve this?

Solve $\cancel{P}\left(1+\frac{r}{2}\right)^{(2)(12)} = 3\cancel{P}$ for r.

$$\Rightarrow \left(1+\frac{r}{2}\right)^{24} = 3$$

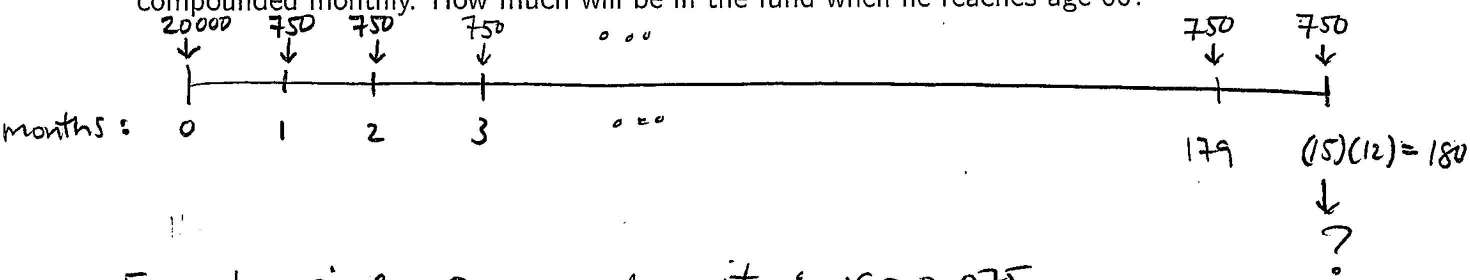
$$1+\frac{r}{2} = 3^{\frac{1}{24}}$$

$$r = 2\left(3^{\frac{1}{24}}-1\right)$$

$$r \doteq 0.0937 \doteq \boxed{9.37\%}$$

Question 8 [10 points]:

(a)[5] Smithers' 45th birthday is on January 1 2012 and he has decided that it is time to get serious about retirement. He will deposit \$20,000 on his upcoming birthday, and then make deposits of \$750 at the end of each month for the next 15 years. All deposits will be made to a fund earning 7.5% interest compounded monthly. How much will be in the fund when he reaches age 60?



For the single 20000 deposit : $r = 0.075$, $n = 12$, $t = 15$, $P = 20000$

For the annuity : $i = \frac{0.075}{12}$, $m = 180$, $P = 750$.

$\therefore$ Value at age 60 is $20000\left(1+\frac{0.075}{12}\right)^{(12)(15)} + 750\left[\frac{\left(1+\frac{0.075}{12}\right)^{180}-1}{\left(\frac{0.075}{12}\right)}\right] \doteq \boxed{\$309{,}723}$

(b)[5] A \$300,000 bank loan will be repaid by making payments at the end of every month. One bank offers 4% interest compounded monthly with a 20 year repayment period, while another bank offers 5% compounded monthly with a 25 year repayment period. Which repayment option results in the lowest monthly payment?

Bank 1

$V_1 = 300{,}000$

$i = \frac{0.04}{12}$

$m = (20)(12) = 240$

$\therefore V_1 = P_1\left[\frac{1-(1+i)^{-m}}{i}\right]$

$P_1 = \frac{i V_1}{1-(1+i)^{-m}}$

$P_1 = \frac{\left(\frac{0.04}{12}\right)(300000)}{\left[1-\left(1+\frac{0.04}{12}\right)^{-240}\right]}$

$P_1 = 1817.94$

Bank 2

$V_2 = 300\,000$

$i = \frac{0.05}{12}$

$m = (25)(12) = 300$

$\therefore P_2 = \frac{\left(\frac{0.05}{12}\right)(300\,000)}{\left[1-\left(1+\frac{0.05}{12}\right)^{-300}\right]}$

$P_2 = 1753.77$

$\therefore$ 5% over 25 yrs results in the lowest monthly payment.

Question 9 [10 points]:

(a)[2] A senior's club has 40 female and 25 male members. An 8 person delegation of 4 female and 4 male members is to be selected to attend a regional meeting. How many different delegations are possible?

number of delegations is $C(40,4)\cdot C(25,4)$

$$= \frac{40!}{36!\,4!}\cdot\frac{25!}{21!\,4!}$$

$$= \boxed{1,156,083,500}$$

(b)[2] A bank machine requires a six digit PIN code where the digits can be $0, 1, 2, \ldots, 9$. How many PIN codes have at least one repeated digit?

$$\begin{pmatrix}\text{number with at}\\ \text{least 1 repeated}\\ \text{digit}\end{pmatrix} = \begin{pmatrix}\text{number of all}\\ \text{possible PIN}\\ \text{codes}\end{pmatrix} - \begin{pmatrix}\text{number with no}\\ \text{repeated digits}\end{pmatrix}$$

$$= 10^6 - (10)(9)(8)(7)(6)(5)$$

$$= \boxed{848,800}$$

(c)[3] How many different ways can 3 red, 4 yellow and 5 blue bulbs be arranged in a string of Christmas tree lights with 12 sockets?

number of ways is $C(12,3)\cdot C(9,4)\cdot C(5,5)$

$$= \frac{12!}{9!\,3!}\cdot\frac{9!}{4!\,5!}\cdot\frac{5!}{0!\,5!}$$

$$= \frac{12!}{3!\,4!\,5!} = \boxed{27,720}$$

(d)[3] Suppose A and B are sets with the property that $n(A \cup B) = n(A \cap B)$. What is $n(A) - n(B)$?

$(A\cap B) \subseteq A \subseteq (A\cup B)$, so $n(A\cap B) \le n(A) \le n(A\cup B)$,

so $n(A\cup B) \le n(A) \le n(A\cup B)$,

so $n(A) = n(A\cup B)$.

Similarly, $n(B) = n(A\cup B)$.

$\boxed{\therefore\ n(A) - n(B) = 0}$

Question 10 [10 points]:

(a)[3] The odds for event E are 1 to 2, and the odds for event F are 2 to 3. If $P(E \cap F) = 0$ determine the odds for $E \cup F$, that is, the odds for E or F.

$$P(E) = \frac{1}{1+2} = \frac{1}{3}$$
$$P(F) = \frac{2}{2+3} = \frac{2}{5}$$

$$P(E \cup F) = P(E) + P(F) - \cancel{P(E \cap F)} \to 0$$
$$= \frac{1}{3} + \frac{2}{5}$$
$$= \frac{11}{15} = \frac{11}{11+4}$$

$\boxed{\therefore \text{ odds for } E \text{ or } F \text{ are } 11 \text{ to } 4.}$

(b)[2] Two balls are drawn, one after the other and without replacement, from a box containing 4 white, 3 green and 2 yellow balls. What is the probability that one ball is white and the other yellow?

$E =$ "one ball white, one yellow"; $n(E) = C(4,1) \cdot C(2,1) = 8$

$S =$ "any two balls"; $n(S) = C(9,2)$.

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{8}{C(9,2)} = \frac{8}{\left(\frac{9!}{7!2!}\right)} = \boxed{\frac{2}{9}}$$

(c)[2] Again, two balls are drawn, one after the other and without replacement, from a box containing 4 white, 3 green and 2 yellow balls. What is the probability that the first ball is white and the second is yellow?

$E =$ "first white, second yellow"; $n(E) = C(4,1) \cdot C(2,1) = 8$

$S =$ "any two balls (order important)"

$n(S) = P(9,2)$.

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{8}{\left(\frac{9!}{7!}\right)} = \boxed{\frac{1}{9}}$$

(d)[3] A person pays \$3 to roll a pair of dice. If the person rolls a "double", meaning ⚀⚀, ⚁⚁, etc, then he wins \$12, otherwise he wins nothing. What is the expected net payoff of such a game?

$A =$ "roll a double"; $P(A) = \frac{6}{36}$; net payoff $p_1 = \$9$.

$\bar{A} =$ "don't roll a double"; $P(\bar{A}) = \frac{30}{36}$; net payoff $p_2 = \$-3$.

$$\therefore E = (9)\left(\frac{6}{36}\right) + (-3)\left(\frac{30}{36}\right) = \boxed{\$1}$$