Additional Problems for Assignment 5

We saw that the formal definition of a continuous function properly characterizes the notion of continuity for familiar functions like $f(x) = x^2$ and $g(x) = 1/x$. That is, to define continuity at a point, the formal $\epsilon\delta$ definition is the 'right one' in some sense. It is interesting to now apply this definition to more unusual functions and see what it tells us about continuity in such cases. For this exercise recall that $\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$

Consider the function $f : [0,1] \to \mathbb{R}$ defined as follows: let $\{r_i\}_{i=1}^{\infty}$ be an enumeration of the rational numbers in the interval $(0,1)$, and define

$$f(x) = \sum_{r_i \le x} \frac{1}{2^i} \ .$$

Here's another way to write the function f: letting

$$\theta_{r_i}(x) = \begin{cases} 1 & \text{if } r_i \le x \\ 0 & \text{if } r_i > x \end{cases} \quad ,$$

f can then be expressed

$$f(x) = \sum_{i=1}^{\infty} \frac{\theta_{r_i}(x)}{2^i} \ .$$

In either case, f is an increasing function on $[0,1]$ that jumps by $1/2^i$ at each rational $r_i \in (0,1)$.

1. What is $f(0)$?
2. What is $f(1)$?
3. Show that f is discontinuous at every rational in $(0,1)$. Hint: for any rational r in $(0,1)$ consider a sequence of rational numbers $\{x_n\}_{n=1}^{\infty}$ that increases to r. Show that $f(x_n) \not\to f(r)$.
4. Suppose $c \in (0,1)$ is irrational. What is $\lim_{n\to\infty} f\left(c + \frac{1}{n}\right) - f\left(c - \frac{1}{n}\right)$? Hint: first, letting $A_n = \left(c - \frac{1}{n}, c + \frac{1}{n}\right]$, note that
$$f\left(c + \frac{1}{n}\right) - f\left(c - \frac{1}{n}\right) = \sum_{r_i \in A_n} \frac{1}{2^i} \ .$$
Let $\epsilon > 0$ be given and select $M \in \mathbb{N}$ so that $\sum_{i>M} \frac{1}{2^i} < \epsilon$. Now think about $\{r_1, \dots, r_M\} \cap A_n$ as n increases.
5. Use the previous exercise to show that f is continuous at every irrational $c \in (0,1)$.