Question 1:

(a)[3] Determine an equation of the line passing through the two points $(3,-4)$ and $(2,1)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-4-1}{3-2}$$

$$= -5$$

$\therefore$ $y - 1 = -5(x-2)$

or $y = -5x + 11$

(b)[3] Determine an equation of the horizontal line passing through the point $(-5,7)$.

$y = 7$

(c)[4] Determine the point of intersection of the following pair of lines:

① $3x + 4y = 2$

② $x - 2y = 1$

② $\Rightarrow$ $x = 1 + 2y$

① $\Rightarrow$ $3(1+2y) + 4y = 2$

$3 + 6y + 4y = 2$

$10y = -1$

$y = \frac{-1}{10}$

$\therefore$ $x = 1 + 2\left(\frac{-1}{10}\right)$

$= 1 - \frac{1}{5}$

$= \frac{4}{5}$

$\therefore$ $(x, y) = \left(\frac{4}{5}, \frac{-1}{10}\right)$

Question 2:

(a)[5] A theatre charges \$8 for each adult ticket while tickets for children cost \$4 each. In one week 2600 tickets were sold resulting in total revenue of \$16,440. Determine the number of adult tickets sold.

Let x = no. of adult tickets
y = no. of children's tickets.

① $x + y = 2600$

② $8x + 4y = 16{,}440$

① $\Rightarrow y = 2600 - x$

② $\Rightarrow 8x + 4(2600 - x) = 16{,}440$

$4x + 10{,}400 = 16{,}440$

$\therefore x = \dfrac{16{,}440 - 10{,}400}{4}$

$x = 1510$

$\therefore$ 1510 adult tickets were sold

(b)[5] The library photocopier has monthly fixed costs of \$160 and it costs the library a further \$0.02 for each copy made. The library charges \$0.06 for each copy made. How many copies must be made each month in order for the library to break even on the photocopier costs?

Let x = number of copies made in a month.

$C = 160 + 0.02x$

$R = 0.06x$

solving $C = R$: $160 + 0.02x = 0.06x$

$160 = 0.04x$

$\therefore x = \dfrac{160}{0.04}$

$= 4000.$

$\therefore$ 4000 copies must be made to break even

Question 3:

(a)[8] Solve the following system of equations using matrix reduction (no credit will be given for using any other method). Clearly state the row operations you are using, and clearly state your final answer.

$$2x + 4y + 2z = 6$$
$$2x + y + z = 16$$
$$x + y + 2z = 9$$

$$\left[\begin{array}{ccc|c} 2 & 4 & 2 & 6 \\ 2 & 1 & 1 & 16 \\ 1 & 1 & 2 & 9 \end{array}\right]$$

$r_1 \leftrightarrow r_3$:
$$\left[\begin{array}{ccc|c} \textcircled{1} & 1 & 2 & 9 \\ 2 & 1 & 1 & 16 \\ 2 & 4 & 2 & 6 \end{array}\right]$$

$R_2 = (-2)r_1 + r_2$:
$R_3 = (-2)r_1 + r_3$:
$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & -1 & -3 & -2 \\ 0 & 2 & -2 & -12 \end{array}\right]$$

$R_2 = (-1)r_2$:
$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & \textcircled{1} & 3 & 2 \\ 0 & 2 & -2 & -12 \end{array}\right]$$

$R_3 = (-2)r_2 + r_3$:
$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -8 & -16 \end{array}\right]$$

$R_3 = \left(\frac{-1}{8}\right)r_3$:
$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$\therefore z = 2$

$y + 3z = 2 \Rightarrow y = 2 - 3(2) = -4$

$x + y + 2z = 9 \Rightarrow x = 9 - (-4) - 2(2) = 9$

$\therefore x = 9,\ y = -4,\ z = 2$

(b)[3] Simplify the following to a single matrix:

$$\begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 7 & 4 & -2 \\ 4 & 3 & -5 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -5 & 13 \\ 36 & 22 & -18 \end{bmatrix} + \begin{bmatrix} -3 & 0 & -3 \\ 0 & -3 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & -5 & 10 \\ 36 & 19 & -24 \end{bmatrix}$$

Question 4:

(a)[4] Determine $\mathbf{A}^{-1}$ where $\mathbf{A}$ is the matrix

$$\begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 4 & 7 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$$

$r_1 \leftrightarrow r_2$: $\left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 4 & 7 & 1 & 0 \end{array}\right]$

$R_2 = (-4)r_1 + r_2$: $\left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -4 \end{array}\right]$

$R_2 = (-1)r_2$: $\left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 4 \end{array}\right]$

$R_1 = (-2)r_2 + r_1$: $\left[\begin{array}{cc|cc} 1 & 0 & 2 & -7 \\ 0 & 1 & -1 & 4 \end{array}\right]$

$\therefore A^{-1} = \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$.

(b)[5] Determine the number of solutions to a system of equations that has the following matrix representation:

$$\left[\begin{array}{cc|c} 1 & 3 & 11 \\ 3 & -4 & -6 \\ 2 & -7 & -17 \end{array}\right]$$

The answer is either no solution, one solution, or infinitely many solutions. State which, with justification.

$R_2 = (-3)r_1 + r_2$:
$R_3 = (-2)r_1 + r_3$: $\left[\begin{array}{cc|c} 1 & 3 & 11 \\ 0 & -13 & -39 \\ 0 & -13 & -39 \end{array}\right]$

$R_3 = (-1)r_2 + r_3$: $\left[\begin{array}{cc|c} 1 & 3 & 11 \\ 0 & -13 & -39 \\ 0 & 0 & 0 \end{array}\right]$

$R_2 = \left(-\frac{1}{13}\right)r_2$: $\left[\begin{array}{cc|c} 1 & 3 & 11 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$

$\therefore$ system has a single solution

Question 5 [10]:

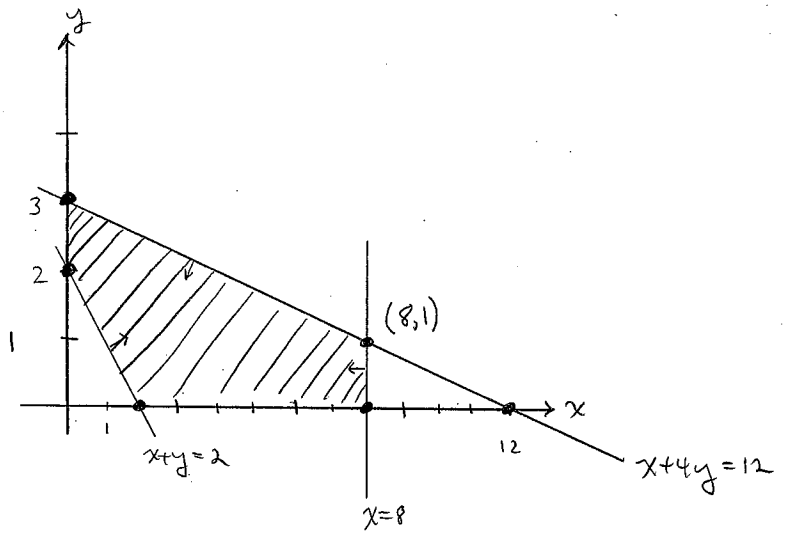
Maximize

$$z = x + 5y$$

subject to the constraints

$$\begin{aligned} x + 4y &\leq 12 \\ x &\leq 8 \\ x + y &\geq 2 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Neatly draw any required graphs and clearly show your work if determining corner points. State a clear conclusion.



corner points: • by inspection: $(0,3), (0,2), (2,0)$ $(8,0)$.

• intersection of $x = 8$ and $x + 4y = 12$ $\therefore 8 + 4y = 12$, $y = \frac{12-8}{4} = 1$, $\therefore (8,1)$

C.P.	$z = x + 5y$
$(0,3)$	$z = 0 + 5(3) = 15$
$(0,2)$	$z = 0 + 5(2) = 10$
$(2,0)$	$z = 2 + 5(0) = 2$
$(8,0)$	$z = 8 + 5(0) = 8$
$(8,1)$	$z = 8 + 5(1) = 13$

$\therefore$ z has a maximum of 15 at $x = 0, y = 3$.

Question 6:

(a)[3] An amount of money invested at an interest rate of r compounded quarterly doubles in 8 years. Determine r. (State your answer as a percentage rounded to 2 decimals.)

$$A\left(1+\frac{r}{4}\right)^{(4)(8)} = 2A$$

$$\therefore \quad r = 4\left(2^{\frac{1}{32}} - 1\right)$$

$$\boxed{r \approx 8.76\%}$$

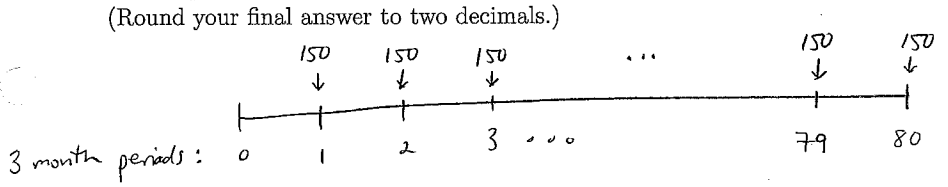
(b)[3] What is the effective rate of interest equivalent to 4.5% compounded semi-annually? (State your answer as a percentage rounded to 2 decimals.)

$$1+R = \left(1+\frac{0.045}{2}\right)^2$$

$$\therefore \quad R = \left(1+\frac{0.045}{2}\right)^2 - 1$$

$$\boxed{R \approx 4.55\%}$$

(c)[4] At the end of each three month period for the next 20 years \$150 is deposited into a fund paying 8% compounded quarterly. What is the accumulated value of the fund at the end of the 20 years? (Round your final answer to two decimals.)



Here $r = 0.08$

$n = 4$

$m = (4)(20) = 80$

$i = \frac{r}{n} = 0.02$

$P = 150$

$$\therefore \quad A = 150\left[\frac{(1+0.02)^{80}-1}{0.02}\right]$$

$$\boxed{A \approx \$29,065.79}$$

Question 7:

(a)[5] A graduating student with \$30,000 in student loans negotiates a ten year repayment plan with his bank. The student will make an initial repayment of \$4000 now and repay the remaining balance with payments made at the end of each month. Determine the required monthly payments if the negotiated loan interest rate is 6% compounded monthly. (Round your final answer to two decimals.)

Amount to be amortized is $V = 30{,}000 - 4000 = \$26{,}000$.

$n = 12$, $r = 0.06$.

$\therefore m = (12)(10) = 120$

$i = \frac{r}{n} = \frac{0.06}{12} = 0.005$

$\therefore V = P\left[\frac{1-(1+i)^{-m}}{i}\right]$

$\Rightarrow P = \frac{iV}{1-(1+i)^{-m}} = \frac{(0.005)(26{,}000)}{1-(1+0.005)^{-120}}$

$P \approx \$288.65$

(b)[5] Lenny and Carl have a plan to retire in ten years time and they each wish to have \$100,000 saved by retirement day. Lenny starts making deposits at the end of each month into a fund earning 5% interest compounded monthly and continues these deposits for the full ten years. Carl waits three years before starting his monthly deposits, but finds a fund which earns 7% interest compounded monthly.

(i) Who has the larger monthly payment, Lenny or Carl?

$A = P\left[\frac{(1+i)^m - 1}{i}\right] \Rightarrow P = \frac{iA}{(1+i)^m - 1}$ where $A = 100{,}000$.

Lenny: $i = \frac{0.05}{12}$, $m = (12)(10) = 120$

$\therefore P_L = \frac{\left(\frac{0.05}{12}\right)(100{,}000)}{\left(1+\frac{0.05}{12}\right)^{120}-1} \approx \$643.99$

Carl: $i = \frac{0.07}{12}$, $m = (7)(12) = 84$.

$\therefore P_C = \frac{\left(\frac{0.07}{12}\right)(100{,}000)}{\left(1+\frac{0.07}{12}\right)^{84}-1} \approx \$925.93$

$\therefore$ Carl's monthly payment is larger

(ii) Who earns more interest, Lenny or Carl?

Lenny: amount of interest $= 100{,}000 - (120)(643.99)$
$= 22{,}721.20$

Carl: amount of interest $= 100\,000 - (84)(925.93)$
$= 22{,}221.88$

$\therefore$ Lenny earns more interest

Question 8:

(a)[3] Two sets M and N are such that $n(M \cup N) = 12$ and $n(M \cap N) = 7$. If M contains 9 elements, how many elements does N contain?

$$n(M \cup N) = n(M) + n(N) - n(M \cap N)$$

$$\therefore n(N) = n(M \cup N) - n(M) + n(M \cap N)$$

$$= 12 - 9 + 7$$

$$= \boxed{10}$$

(b)[4] Let $U = \{1,2,3,4,5,6,7,8\}$, $A = \{2,4,6,8\}$, $B = \{1,3,5,7,8\}$ and $C = \{1,4,7\}$.

(i) Determine $A \cap \overline{C}$

$$\overline{C} = \{2,3,5,6,8\}$$

$$\therefore A \cap \overline{C} = \boxed{\{2,6,8\}}$$

(ii) Determine $\overline{(B \cup C)} \cap A$.

$$B \cup C = \{1,3,4,5,7,8\}$$

$$\overline{B \cup C} = \{2,6\}$$

$$\therefore (\overline{B \cup C}) \cap A = \boxed{\{2,6\}}$$

(c)[3] What is the coefficient of x^3 in the expansion of $(2x-1)^5$?

$$\binom{5}{2}(2x)^{5-2}(-1)^2 = \frac{5!}{3!\,2!} \cdot 2^3 (-1)^2 x^3$$

$$= 80x^3$$

$\therefore$ the coefficient is 80

Question 9:

(a)[3] A class has 9 boys and 7 girls. How many different committees of 5 students can be formed if the committees must contain exactly 3 boys?

$$C(9,3)\,C(7,2) = \frac{9!}{6!\,3!} \cdot \frac{7!}{5!\,2!}$$

$$= \boxed{1764}$$

(b)[3] Again a class has 9 boys and 7 girls. How many different committees of 5 students can be formed if the committees must contain at least one boy?

$$C(16,5) - C(7,5) = \frac{16!}{11!\,5!} - \frac{7!}{2!\,5!}$$

$$= \boxed{4347}$$

(c)[4] How many different 4 letter codes can be formed using the letters P, Q, R, S if

(i) each letter can appear only once?

$$P(4,4) = 4! = \boxed{24}$$

(ii) the letter P appears two times, while the letters Q, R, S may appear at most once?

$$C(4,2)\,P(3,2) = \frac{4!}{2!\,2!} \cdot \frac{3!}{1!}$$

$$= \boxed{36}$$

Question 10:

(a)[3] From a box of 10 red and 5 green balls four balls are drawn without replacement. Determine the probability of obtaining exactly two green balls. (Round your final answer to three decimals.)

E = "exactly 2 green"

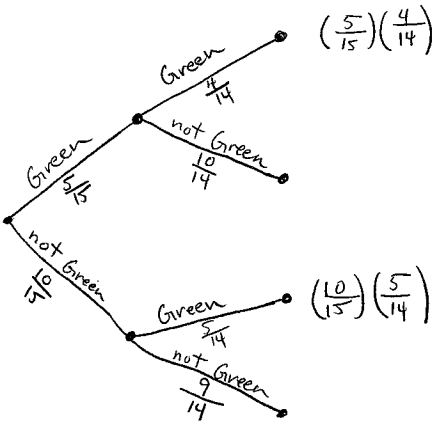
S = "all possible groups of 4 balls"

$$P(E) = \frac{n(E)}{n(S)} = \frac{C(5,2)\,C(10,2)}{C(15,4)}$$

$$= \frac{\left(\frac{5!}{3!2!}\right)\left(\frac{10!}{8!2!}\right)}{\left(\frac{15!}{11!4!}\right)}$$

$$\approx \boxed{0.330}$$

(b)[3] From a box of 10 red and 5 green balls two balls are drawn, one after the other, without replacement. Determine the probability that the second ball is green. (Round your final answer to three decimals.)



$$\therefore P(2^{nd} \text{ ball green}) = \left(\frac{5}{15}\right)\left(\frac{4}{14}\right) + \left(\frac{10}{15}\right)\left(\frac{5}{14}\right)$$

$$\approx \boxed{0.333}$$

(c)[4] If the odds for A are 2 to 1, the odds against B are 3 to 1, and the odds for A and B are 1 to 2, determine the odds for A or B. ~~(Round your final answer to three decimals.)~~

$$P(A) = \frac{2}{2+1} = \frac{2}{3}$$

$$P(B) = \frac{1}{3+1} = \frac{1}{4}$$

$$P(A \cap B) = \frac{1}{1+2} = \frac{1}{3}$$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{2}{3} + \frac{1}{4} - \frac{1}{3}$$

$$= \frac{8+3-4}{12}$$

$$= \frac{7}{7+5}$$

$\therefore$ odds for A or B are 7 to 5.