**Question 1:**

**(a)[5 points]** The function $f(x)$ has the following values:

| $x$ | 0 | 1/2 | 1 | 3/2 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | $-4$ | 2 | $-2$ | $a$ | $-6$ |

Notice in this table that $f(3/2) = a$. Using all of the data above and Simpson's Rule resulted in the approximation $\int_0^2 f(x)\,dx \approx 1$. Determine the value of $a$.

$n = 4$,
$\Delta x = \frac{1}{2}$

$$\frac{\Delta x}{3}\left[f(0) + 4f(\tfrac{1}{2}) + 2f(1) + 4f(\tfrac{3}{2}) + f(2)\right] = 1$$

$$\frac{1}{6}\left[-4 + (4)(2) + (2)(-2) + (4)(a) - 6\right] = 1$$

$$4a - 6 = 6$$

$$4a = 12$$

$$\boxed{a = 3}$$

**(b)[5 points]** Suppose Simpson's rule on 12 subintervals is used to approximate $\int_0^{12} g(x)\,dx$, where $g(x) = x^3 + px + q$ for some constants $p$ and $q$. Determine an error bound for the approximation. (Recall, the error in using Simpson's Rule to approximate $\int_a^b g(x)\,dx$ is at most $\frac{K(b-a)^5}{180n^4}$, where $|g^{(4)}(x)| \le K$ on $[a, b]$.) The answer is simple, and surprising.

$g'(x) = 3x^2 + p$

$g''(x) = 6x$

$g'''(x) = 6$

$g^{(4)}(x) = 0$

$\therefore K = 0$

$\therefore$ Error in approximation $S_{12} \approx \int_0^{12} g(x)\,dx$ is zero, i.e.

$$\left|S_{12} - \int_0^{12} g(x)\,dx\right| = 0.$$

**Question 2:**

**(a)[5 points]** Evaluate the improper integral. Clearly show all steps including any required limits, and state, based on your answer, whether the integral converges or diverges:

$$\int_0^\infty \frac{x}{(1+2x^2)^{3/2}}\,dx$$

For $I = \int \frac{x}{(1+2x^2)^{3/2}}\,dx$, let $u = 1+2x^2$, $du = 4x\,dx$,

$$\therefore I = \frac{1}{4}\int u^{-3/2}\,du = \frac{1}{4}\frac{u^{-1/2}}{(-1/2)} = -\frac{1}{2}\frac{1}{(1+2x^2)^{1/2}}$$

$$\therefore \int_0^\infty \frac{x}{(1+2x^2)^{3/2}}\,dx = \lim_{b\to\infty}\int_0^b \frac{x}{(1+2x^2)^{3/2}}\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{1}{2}\frac{1}{(1+2x^2)^{1/2}}\right]_0^b$$

$$= \lim_{b\to\infty}\left[-\frac{1}{2}\frac{1}{(1+2b^2)^{1/2}} + \frac{1}{2}\right]$$

$$= \boxed{\frac{1}{2}} \leftarrow \text{so integral } \underline{\text{converges}}.$$

**(b)[5 points]** Use the Comparison Test to determine if $\int_1^\infty \frac{x^2}{x^4+e^x}\,dx$ converges or diverges.

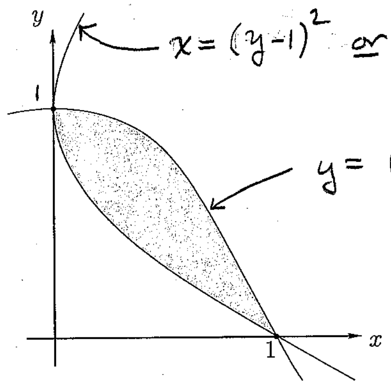
For $x \geq 1$, $0 \leq \frac{x^2}{x^4+e^x} \leq \frac{x^2}{x^4} = \frac{1}{x^2}$

Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges*, by the Comparison Test so must $\int_1^\infty \frac{x^2}{x^4+e^x}\,dx$.

* $\int_1^\infty \frac{1}{x^2}\,dx$ is of type $\int_1^\infty \frac{1}{x^p}\,dx$ where $p>1$

**Question 3:**

**(a)[5 points]** Determine the area of region bounded by the two curves $y = 1 - x^4$ and $x = (y-1)^2$:




$x = (y-1)^2$ or $y = 1 - \sqrt{x}$

$y = 1 - x^4$ or $x = (1-y)^{\frac{1}{4}}$

Option 1:

$$A = \int_0^1 (1-x^4) - (1 - x^{\frac{1}{2}})\, dx$$

$$= \left[\frac{2}{3} x^{\frac{3}{2}} - \frac{x^5}{5}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{5}$$

$$= \frac{7}{15}$$

Option 2:

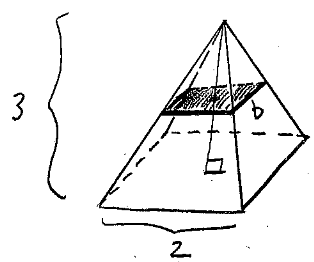
$$A = \int_0^1 (1-y)^{\frac{1}{4}} - (y-1)^2\, dy$$

$$= \left[-\frac{4}{5}(1-y)^{\frac{5}{4}} - \frac{(y-1)^3}{3}\right]_0^1$$

$$= (0 - 0) - \left(-\frac{4}{5} + \frac{1}{3}\right)$$

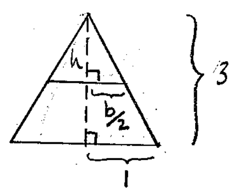
$$= \frac{7}{15}$$

**(b)[5 points]** A pyramid has height 3 m and a square base of side length 2 m. Determine the volume of the pyramid.




Slice the solid parallel to base a distance h from the top; resulting Cross-section is a square of side length b.

Viewed from the side:




By similar triangles: $\frac{h}{\left(\frac{b}{2}\right)} = \frac{3}{1}$

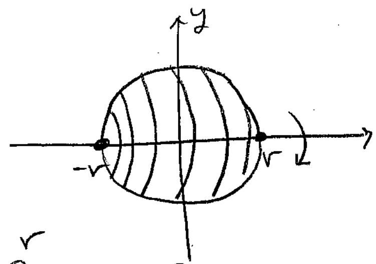
$\therefore\ b = \left(\frac{2}{3}h\right)$

$\therefore\ A(h) = \text{cross-sectional area} = \left(\frac{2}{3}h\right)^2 = \frac{4}{9}h^2$

$$\therefore V = \int_{h=0}^{h=3} \frac{4}{9}h^2\, dh = \frac{4}{9}\left[\frac{h^3}{3}\right]_0^3 = \frac{4}{9} \cdot \frac{27}{3} = \boxed{4\ m^3}$$

**Question 4:**

**(a)[5 points]** The top half of a circle of radius $r$ is described by the curve $y = \sqrt{r^2 - x^2}$, $-r \le x \le r$. Rotate this curve about the $x$-axis and determine volume of the resulting sphere.



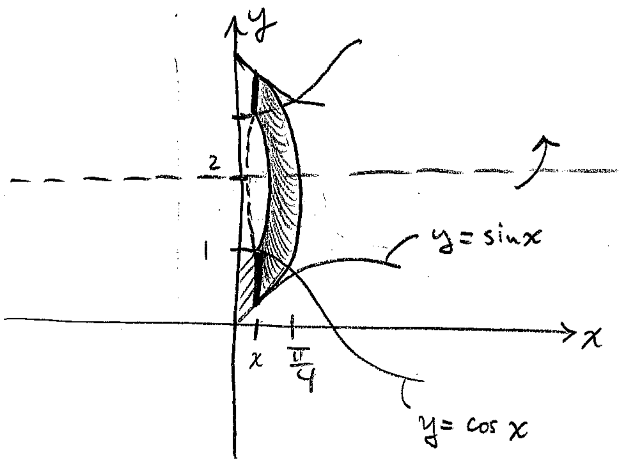

$$V = \int_{x=-r}^{r} \pi\left(\sqrt{r^2 - x^2}\right)^2 dx = \int_{-r}^{r} \pi\,(r^2 - x^2)\,dx$$

$$= \pi\left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r}$$

$$= \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right]$$

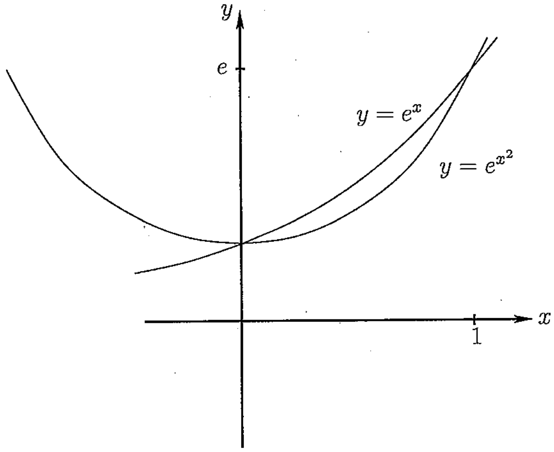
$$= \frac{4}{3}\pi r^3.$$

**(b)[5 points]** The region bounded by $y = \cos x$, $y = \sin x$, $x = 0$ and $x = \pi/4$ is rotated about the line $y = 2$. Set up but do not evaluate an integral representing the volume of the resulting solid. Disks would work best here.

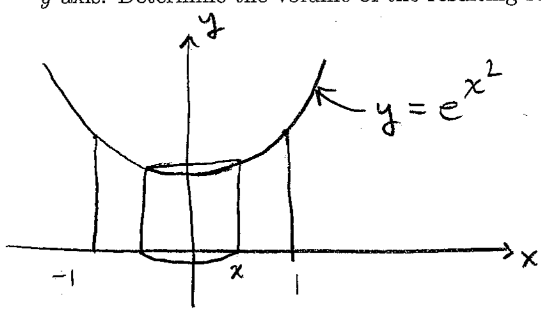



$$V = \int_{0}^{\frac{\pi}{4}} \pi\left[(2 - \sin x)^2 - (2 - \cos x)^2\right] dx.$$

**Question 5:** This question deals with the functions $y = e^{x^2}$ and $y = e^x$. The following may be helpful:




**(a)[5 points]** The region bounded by $y = e^{x^2}$, the $x$-axis, $x = 0$ and $x = 1$ is rotated about the $y$-axis. Determine the volume of the resulting solid. Cylindrical shells would be best here.

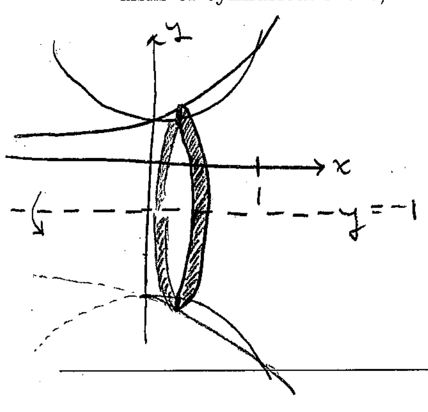



$$V = \int_0^1 2\pi x e^{x^2}\,dx$$

$$= \pi \left[ e^{x^2} \right]_0^1$$

$$= \pi (e - 1)$$

**(b)[5 points]** The region bounded between $y = e^{x^2}$ and $y = e^x$ is rotated about the line $y = -1$. Set up but do not evaluate an integral representing the volume of the resulting solid. Use disks or cylindrical shells, whichever you prefer.




$$V = \int_0^1 \pi \left[ (e^x+1)^2 - (e^{x^2}+1)^2 \right] dx.$$