Question 1:

(a) [6 points] Water pours into a tank at a rate of $r(t) = 4 + t^2$ L/min where $t \geq 0$. How much water enters the tank during the first two minutes? State units with your answer.

Recall: the integral of rate of change gives total change!

Let $V(t)$ = volume of water in tank at time t.

$\therefore V'(t) = 4 + t^2$

$$\therefore V(2) - V(0) = \int_0^2 4 + t^2 \, dt = \left[4t + \frac{t^3}{3}\right]_0^2$$

$$= 8 + \frac{8}{3}$$

$$= \boxed{\frac{32}{3} \text{ L}}$$

(b) [7 points] Determine the average value of $f(x) = e^{\sin(x)} \cos(x)$ over the interval $[0, \pi/2]$.

$$f_{ave} = \frac{1}{\frac{\pi}{2} - 0} \int_0^{\frac{\pi}{2}} e^{\sin(x)} \cos(x) \, dx$$

$u = \sin(x)$, $du = \cos(x)\,dx$

$x = 0 \Rightarrow u = \sin(0) = 0$

$x = \frac{\pi}{2} \Rightarrow u = \sin(\frac{\pi}{2}) = 1$

$$\therefore f_{ave} = \frac{2}{\pi} \int_0^1 e^u \, du$$

$$= \frac{2}{\pi} \left[e^u\right]_0^1$$

$$= \boxed{\frac{2(e-1)}{\pi}}$$

Question 2 [10 points]: Evaluate $\int \arccos(x)\,dx = I$

$$u = \arccos(x), \quad dv = dx$$
$$du = \frac{-1}{\sqrt{1-x^2}}\,dx, \quad v = x$$

$$\therefore\ I = \int u\,dv = uv - \int v\,du$$

$$= x\arccos(x) - \int x\frac{(-1)}{\sqrt{1-x^2}}\,dx$$

$$= x\arccos(x) - \frac{1}{2}\int \underbrace{\frac{1}{\sqrt{1-x^2}}(-2)x\,dx}_{\substack{u = 1-x^2 \\ du = -2x\,dx}}$$

$$= x\arccos(x) - \frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

$$= x\arccos(x) - \cancel{\left(\tfrac{1}{2}\right)}\frac{u^{1/2}}{\cancel{(1/2)}} + C$$

$$= \boxed{x\arccos(x) - \sqrt{1-x^2} + C}$$

Question 3 [10 points]: Evaluate $\int \frac{x^3}{\sqrt{x^2+4}}\,dx = I$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$$\therefore\ I = \int \frac{8\tan^3\theta}{\sqrt{4\tan^2\theta+4}}\,2\sec^2\theta\,d\theta$$

$$= \int \frac{8\tan^3\theta\,\cancel{2}\sec^{\cancel{2}}\theta}{\cancel{2}\cancel{\sec\theta}}\,d\theta$$

$$= 8\int \tan^2\theta\ \sec\theta\tan\theta\,d\theta$$

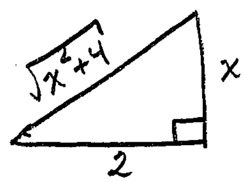
$$= 8\int (\sec^2\theta-1)\sec\theta\tan\theta\,d\theta$$

$u = \sec\theta$

$du = \sec\theta\tan\theta\,d\theta$

$x = 2\tan\theta,$

$\therefore \tan\theta = \frac{x}{2}$:



$$\therefore\ I = 8\int u^2-1\,du$$

$$= 8\left(\frac{u^3}{3}-u\right)+C$$

$$= 8\left[\frac{\sec^3\theta}{3}-\sec\theta\right]+C$$

$$= 8\left[\frac{1}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3-\frac{\sqrt{x^2+4}}{2}\right]+C$$

$$= \boxed{\frac{1}{3}(x^2+4)^{3/2}-4\sqrt{x^2+4}+C}$$

Question 4 [10 points]: Evaluate $\int \frac{3x-5}{x^2+5x-14}\,dx = I$

$$\frac{3x-5}{x^2+5x-14} = \frac{3x-5}{(x+7)(x-2)} = \frac{A}{x+7} + \frac{B}{x-2}$$

$$= \frac{Ax-2A+Bx+7B}{(x+7)(x-2)}$$

$$= \frac{(A+B)x-2A+7B}{(x+7)(x-2)}$$

$$\left.\begin{aligned} \therefore\ A+B &= 3 \\ -2A+7B &= -5 \end{aligned}\right\} \quad \therefore\ B = 3-A$$

$$-2A+7(3-A) = -5$$

$$-9A = -5-21$$

$$A = \frac{26}{9}$$

$$\therefore\ B = 3-\frac{26}{9}$$

$$= \frac{27-26}{9}$$

$$= \frac{1}{9}$$

$$\therefore\ I = \int \frac{\left(\frac{26}{9}\right)}{x+7} + \frac{\left(\frac{1}{9}\right)}{x-2}\,dx$$

$$= \boxed{\frac{26}{9}\ln|x+7| + \frac{1}{9}\ln|x-2| + C}$$

Question 5 [7 points]: Find a function f and the value of the constant a such that

$$2\int_a^x f(t)\,dt = 2\sin(x) - 1$$

$$\frac{d}{dx}\left[2\int_a^x f(t)\,dt\right] = \frac{d}{dx}\left[2\sin(x)-1\right]$$

$$\therefore \cancel{2} f(x) = \cancel{2}\cos(x)$$

$$\therefore f(x) = \cos(x)$$

$$\therefore 2\int_a^x \cos(t)\,dt = 2\sin(x) - 1$$

$$2\left[\sin(t)\right]_a^x = 2\sin(x) - 1$$

$$2\sin(x) - 2\sin(a) = 2\sin(x) - 1$$

$$\therefore -2\sin(a) = -1$$

$$\sin(a) = \frac{1}{2}$$

$$a = \frac{\pi}{6}.$$

$$\boxed{\therefore f(x) = \cos(x) \text{ and } a = \frac{\pi}{6}}$$