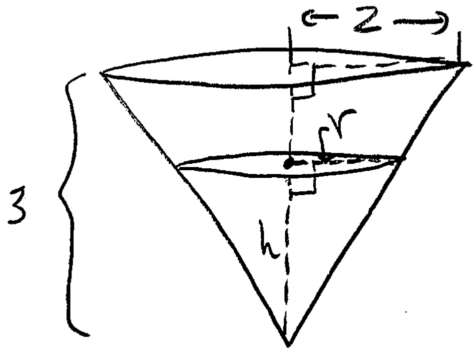
Question 1 [10 points]: A water tank has the shape of an inverted cone of top radius 2 m and height 3 m. Water is leaking from the tank. When the water depth is 2 m it is decreasing at 1/2 m/min. At what rate is the volume of water in the tank changing at that same instant? State units with your answer. Recall that the volume of a cone is given by $V = (\pi/3)r^2h$.



When $h = 2$ m, $\frac{dh}{dt} = -\frac{1}{2}\ \frac{m}{min}$

Find $\frac{dV}{dt}$ when $h = 2$ m.

By similar triangles $\frac{r}{h} = \frac{2}{3} \Rightarrow r = \frac{2}{3}h$

$\therefore V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2}{3}h\right)^2 h = \frac{4\pi}{27}h^3$

$\therefore \frac{dV}{dt} = \frac{4\pi}{27}\,3h^2\,\frac{dh}{dt}$

When $h = 2$:

$$\frac{dV}{dt} = \frac{4\pi}{27} \cdot 3(2)^2 \cdot \left(-\frac{1}{2}\right)$$

$$= -\frac{8\pi}{9}\ \frac{m^3}{min}$$

$\therefore$ The volume is decreasing by $\frac{8\pi}{9}\ \frac{m^3}{min}$.

Question 2:

(a)[5 points] Give the linearization $L(x)$ of $f(x) = x\ln(x^2-3)$ at $a = 2$.

$$f(x) = x\ln(x^2-3) \;;\; f(a) = f(2) = \cancel{2\cdot\ln(2^2-3)} \to 0$$

$$f'(x) = \ln(x^2-3) + (x)\cdot\left(\frac{1}{x^2-3}\right)\cdot(2x)$$

$$f'(2) = \cancel{\ln(2^2-3)} \to 0 + (2)\left(\cancel{\frac{1}{2^2-3}} \to 1\right)(4) = 8$$

$$\therefore L(x) = f(a) + f'(a)(x-a)$$

$$= 0 + 8(x-2)$$

$$= 8x - 16.$$

(b)[5 points] Use a linear approximation to estimate $\frac{1}{\sqrt{99}}$.

Here $f(x) = x^{-\frac{1}{2}}$, $a = 100$.

$$f(a) = f(100) = \frac{1}{\sqrt{100}} = \frac{1}{10}$$

$$f'(x) = -\frac{1}{2}x^{-3/2} = \frac{-1}{2\ x^{3/2}}$$

$$f'(a) = f'(100) = \frac{-1}{2(100)^{3/2}} = \frac{-1}{2\cdot 1000} = \frac{-1}{2000}$$

$$\therefore L(x) = \frac{1}{10} - \frac{1}{2000}(x-100)$$

$$\therefore \frac{1}{\sqrt{99}} \approx L(99) = \frac{1}{10} - \frac{1}{2000}(99-100) = \frac{1}{10} + \frac{1}{2000}$$

$$= \frac{201}{2000}$$

Question 3:

(a)[3 points] Differentiate: $g(x) = \log_7(x + \sin x)$.

$$g'(x) = \frac{1}{(x+\sin x)\ln 7} \cdot (1+\cos x)$$

(b)[3 points] Differentiate: $y = 3^{x^3 \sec x}$.

$$y' = 3^{x^3 \sec x} \cdot \ln 3 \cdot \left[3x^2 \sec x + x^3 \sec x \tan x\right].$$

(c)[4 points] Evaluate and simplify $f''(0)$ if $f(x) = \ln\left[e^x + \ln(1+x)\right]$.

$$f'(x) = \frac{1}{e^x + \ln(1+x)} \cdot \left[e^x + \frac{1}{1+x}\right] = \frac{e^x + (1+x)^{-1}}{e^x + \ln(1+x)}$$

$$f''(x) = \frac{\left[e^x + \ln(1+x)\right]\left[e^x - (1+x)^{-2}\right] - \left[e^x + (1+x)^{-1}\right]\left[e^x + \frac{1}{1+x}\right]}{\left[e^x + \ln(1+x)\right]^2}$$

$$f''(0) = \frac{\left[e^0 + \cancel{\ln(1+0)}^{\,0}\right]\left[\cancel{e^0 - (1+0)^{-2}}^{\,0}\right] - \left[e^0 + (1+0)^{-1}\right]\left[e^0 + \frac{1}{1+0}\right]}{\left[e^0 + \cancel{\ln(1+0)}^{\,0}\right]^2}$$

$$= -4$$

Question 4:

(a)[5 points] Use logarithmic differentiation to compute $f'(x)$ if $f(x) = \sqrt{x}e^{x^3}(x^2-1)^5$.

$$y = x^{1/2} e^{x^3} (x^2-1)^5$$

$$\ln y = \ln\left[x^{1/2} e^{x^3} (x^2-1)^5\right]$$

$$\ln y = \frac{1}{2}\ln x + x^3 + 5\ln(x^2-1)$$

$$\frac{1}{y}y' = \frac{1}{2x} + 3x^2 + \frac{5}{x^2-1}\cdot(2x)$$

$$\therefore\ y' = x^{1/2}e^{x^3}(x^2-1)^5\left[\frac{1}{2x} + 3x^2 + \frac{10x}{x^2-1}\right]$$

(b)[5 points] Use implicit differentiation to determine y' if $e^x = \ln(x^2+y^2)$.

$$e^x = \ln(x^2+y^2)$$

$$\frac{d}{dx}\left[e^x\right] = \frac{d}{dx}\left[\ln(x^2+y^2)\right]$$

$$e^x = \frac{1}{x^2+y^2}(2x + 2yy')$$

$$e^x(x^2+y^2) = 2x + 2yy'$$

$$y' = \frac{e^x(x^2+y^2) - 2x}{2y}.$$

Question 5: For this question use $f(x) = 3x^{2/3} - x$. State clear conclusions for each of the following:

(a)[**5 points**] Determine the intervals of increase and decrease of $f(x)$.

$$f'(x) = 3\left(\tfrac{2}{3}\right)x^{-\frac{1}{3}} - 1 = \frac{2}{x^{1/3}} - 1 = \frac{2 - x^{1/3}}{x^{1/3}}.$$

- $f'(x) = 0$ at $x = 8$
- $f'(x)$ does not exist at $x = 0$.

	-1	0	1	8	27
$f'(x) = \dfrac{2 - x^{1/3}}{x^{1/3}}$:	$-$	NA	$+$	0	$-$
$f(x) = 3x^{2/3} - x$:	$\searrow$		$\nearrow$		$\searrow$

$\therefore$ f is decreasing on $(-\infty, 0)$ and $(8, \infty)$.
f is increasing on $(0, 8)$.

(b)[**5 points**] Determine the intervals of concavity of $f(x)$.

$$f''(x) = \frac{d}{dx}\left[2x^{-\frac{1}{3}} - 1\right] = -\frac{2}{3}x^{-\frac{4}{3}} = -\frac{2}{3}\,\frac{1}{x^{4/3}}.$$

- $f''(x) = 0$? no such x
- $f''(x)$ does not exist at $x = 0$.

	-1	0	8
$f''(x) = -\frac{2}{3}\frac{1}{x^{4/3}}$:	$-$	NA	$-$
$f(x)$:	$\frown$		$\frown$

$\therefore$ f is concave down on $(-\infty, 0)$ and $(0, \infty)$.