Question 1 [10 points]: Use the <u>definition of the derivative</u> to determine $f'(x)$ if $f(x) = \frac{3}{x^2-2}$. (No credit will be given if $f'(x)$ is found using derivative rules, though you may check your answer using the rules.)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{3}{(x+h)^2-2} - \frac{3}{x^2-2}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{3(x^2-2) - 3[(x+h)^2-2]}{[(x+h)^2-2][x^2-2]}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\cancel{3x^2}\cancel{-6}\cancel{-3x^2}-6xh-3h^2\cancel{+6}}{[(x+h)^2-2][x^2-2]}\right]$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\left[\frac{\cancel{h}(-6x-3h)}{[(x+h)^2-2][x^2-2]}\right]$$

$$= \boxed{\frac{-6x}{[x^2-2]^2}}$$

<u>Check</u>: $\frac{d}{dx}\left[\frac{3}{x^2-2}\right] = \frac{d}{dx}\left[3(x^2-2)^{-1}\right]$

$$= -3(x^2-2)^{-2}(2x)$$

$$= \frac{-6x}{(x^2-2)^2}$$

Question 2:

(a)[5 points] Compute $g''(1)$ if $g(x) = \dfrac{2\sqrt{x} - \pi x}{\sqrt[3]{x^2}}$. $= \dfrac{2x^{1/2} - \pi x}{x^{2/3}} = 2x^{-1/6} - \pi x^{1/3}$

$$g'(x) = -\frac{1}{3} x^{-\frac{7}{6}} - \frac{\pi}{3} x^{-\frac{2}{3}}$$

$$g''(x) = \frac{7}{18} x^{-\frac{13}{6}} + \frac{2\pi}{9} x^{-\frac{5}{3}}$$

$$\therefore \quad g''(1) = \frac{7}{18} + \frac{2\pi}{9} = \boxed{\frac{7+4\pi}{18}}$$

(b)[5 points] Determine an equation of the tangent line to the curve $y = \dfrac{\sin(\theta)}{2} - \dfrac{2}{\cos(\theta)}$ at the point where $\theta = \pi$.

At $\theta = \pi$, $y = \cancel{\dfrac{\sin(\pi)}{2}} \to 0 - \dfrac{2}{\cos(\pi)} = 2.$

$y' = \dfrac{\cos(\theta)}{2} + \dfrac{2}{\cos^2\theta}(-\sin\theta)$; $y'\Big|_{\theta=\pi} = \dfrac{-1}{2} + 0 = \dfrac{-1}{2}$

$$\therefore \quad y - y_0 = m(\theta - \theta_0)$$

$$\boxed{y - 2 = \frac{-1}{2}(\theta - \pi)}$$

Question 3:

(a)[3 points] Differentiate: $q(t) = -(5t^3 + t)\sec(t)$.

$$\boxed{q'(t) = -(15t^2+1)\sec(t) - (5t^3+t)\sec(t)\tan(t)}$$

(b)[3 points] Differentiate: $y = \dfrac{x}{\left(x + \frac{c}{x}\right)}$ where c is a constant.

$$y = \frac{x}{x+\frac{c}{x}} \cdot \frac{x}{x} = \frac{x^2}{x^2+c}$$

$$\therefore \boxed{y' = \frac{(x^2+c)(2x) - (x^2)(2x)}{(x^2+c)^2}}$$

(c)[4 points] Evaluate and simplify $f'(1)$ if $f(x) = \dfrac{e^x}{x^e}$.

$$f'(x) = \frac{x^e e^x - e^x e x^{e-1}}{x^{2e}}$$

$$\therefore f'(1) = \frac{1^e \cdot e^1 - e^1 \cdot e \cdot 1^{e-1}}{1^{2e}}$$

$$= \boxed{e - e^2}$$

Question 4:

(a)[3 points] Differentiate: $y = \cos(x^2)\sin^2(x)$.

$$\boxed{y' = -\sin(x^2)(2x)\sin^2(x) + \cos(x^2)\,2\sin(x)\cos(x)}$$

(b)[3 points] Differentiate: $h(x) = \csc(1/x) - \tan(3e^x)$.

$$\boxed{h'(x) = -\csc\left(\tfrac{1}{x}\right)\cot\left(\tfrac{1}{x}\right)\left(\tfrac{-1}{x^2}\right) - \sec^2(3e^x)(3e^x)}$$

(c)[4 points] Compute $\frac{dy}{dx}$: $y = \sqrt{\cos^3(\sqrt{x})}$. $= \left(\left[\cos(x^{\frac{1}{2}})\right]^3\right)^{\frac{1}{2}} = \left[\cos(x^{\frac{1}{2}})\right]$

$$\therefore \boxed{\frac{dy}{dx} = \frac{3}{2}\left[\cos(x^{\frac{1}{2}})\right]^{\frac{1}{2}} \cdot \left(-\sin(x^{\frac{1}{2}})\right) \cdot \frac{1}{2}x^{-\frac{1}{2}}}$$

Question 5:

(a) [**5 points**] Determine y' by implicit differentiation: $y \sin(x^2) = x \sin(y^2)$.

$$\frac{d}{dx}\left[y \sin(x^2)\right] = \frac{d}{dx}\left[x \sin(y^2)\right]$$

$$y' \sin(x^2) + y \cos(x^2)(2x) = \sin(y^2) + x\cos(y^2)(2yy')$$

$$y'\left[\sin(x^2) - 2xy\cos(y^2)\right] = \sin(y^2) - 2xy\cos(x^2)$$

$$\therefore \boxed{y' = \frac{\sin(y^2) - 2xy\cos(x^2)}{\sin(x^2) - 2xy\cos(y^2)}}$$

(b) [**5 points**] Determine all values of x at which tangent lines to the graph of $f(x) = e^{\frac{x^3}{3} - x^2 - 8x}$ are horizontal.

Solve $f'(x) = 0$

$$\underbrace{e^{\frac{x^3}{3} - x^2 - 8x}}_{\neq 0} \cdot (x^2 - 2x - 8) = 0$$

$$\therefore \quad x^2 - 2x - 8 = 0$$

$$(x-4)(x+2) = 0$$

$$\therefore \boxed{x = 4, \quad x = -2}$$